MATB41 Review Seminar

Exam '08 - Q1

Evaluate $\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2}$ or show that it does not exist.

$$= \lim_{(x,y)\to(0,0)} \frac{(\cos xy -1)(\cos xy+1)}{x^2y^2(\cos xy+1)} = \lim_{(x,y)\to(0,0)} \frac{-\sin^2 xy}{x^2y^2(\cos xy+1)} = \lim_{(x,y)\to(0,0)} -\left(\frac{\sin(xy)}{xy}\right)^2\left(\frac{1}{\cos xy+1}\right) = -(1)\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}.$$

TT '14 - Q10.

Give the 4th degree Taylor polynomial about the origin of $f(x,y) = e^{x^2}\cos(xy)$.

$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}, \ |t|<\infty \quad \to e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \cdots, \ |x|<\infty$

$\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}, \ |t|<\infty \quad \to \cos(xy) = 1 - \frac{x^2y^2}{2!} + \frac{x^4y^4}{4!} - \cdots, \ |x,y|<\infty$

$T = \left(1 + x^2 + \frac{x^4}{2!} + \cdots\right)\left(1 - \frac{x^2y^2}{2!} + \frac{x^4y^4}{4!} - \cdots\right) \qquad \therefore T_4 = 1 + x^2 + \frac{x^4}{2} - \frac{x^2y^2}{2}$

TT '15 - Q5b.

Find an equation for the tangent plane at the point $(-3,1,0)$ to the graph of $z=f(x,y)$ define implicitly by $x(y^2+z^2)+ye^{xz} = -2$.

$g(x,y,z) = x(y^2+z^2) + ye^{xz} + 2$

$\nabla g = (y^2+z^2+yze^{xz},\ 2xy + e^{xz},\ 2xz + xye^{xz})$

$\nabla g(-3,1,0) = (1,-5,-3)$

$x - 5y - 3z = k$
$-3-5 = k = -8$

$\therefore$ Eq of tangent plane $x-5y-3z = -8$

TT '12 - Q5c

Find the critical points of $f(x,y,z) = 2x^2 + 2xz + y^2 + 4y + yz$

$\nabla f(x,y,z) = (4x+2z,\ 2y+4+z,\ 2x+y)$

$$\begin{cases} 4x+2z=0 & ① \\ 2y+4+z=0 & ② \\ 2x+y=0 & ③ \end{cases}$$

① $z=-2x$ ③ $y=-2x$

② $2(-2x)+4-2x=0$
$4=6x$
$x = 2/3$

① $z = -4/3$
③ $y = -4/3$

$\therefore$ CPs $\left(2/3, -4/3, -4/3\right)$

Find and classify all critical points of $f(x,y) = x+y+1/x+4/y$; $x,y\neq 0$

$\nabla f = (1 - 1/x^2,\ 1-4/y^2)$

$$Hf = \begin{pmatrix} 2/x^3 & 0 \\ 0 & 8/y^3 \end{pmatrix}$$

$$\begin{cases} 1 - 1/x^2 = 0 \\ 1 - 4/y^2 = 0 \end{cases} \to \begin{cases} x = \pm 1 \\ y = \pm 2 \end{cases}$$

$\therefore$ CPs $(1,2), (1,-2), (-1,2), (-1,-2)$

$Hf(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \to +\ + \quad \therefore$ min

$Hf(1,-2) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \to +\ - \quad \therefore$ saddle

$Hf(-1,2) = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \to -\ + \quad \therefore$ saddle

$Hf(-1,-2) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \to -\ - \quad \therefore$ max.

Exam '08 - Q9

The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

$h(x,y,z,\lambda_1,\lambda_2) = \underbrace{x^2+y^2+z^2}_{\text{distance sq.}} - \lambda_1(x+y+z-1) - \lambda_2(x^2+y^2-1)$

$$\begin{cases} h_x = 2x - \lambda_1 - 2x\lambda_2 = 0 \to \lambda_1 = 2x(1-\lambda_2) = 2z \\ h_y = 2y - \lambda_1 - 2y\lambda_2 = 0 \to \lambda_1 = 2y(1-\lambda_2) = 2z \\ h_z = 2z - \lambda_1 = 0 \to \lambda_1 = 2z \\ h_{\lambda_1} = -(x+y+z-1) = 0 \\ h_{\lambda_2} = -(x^2+y^2-1) = 0 \end{cases}$$

$$\begin{cases} z = x(1-\lambda_2) \\ z = y(1-\lambda_2) \end{cases}$$

$\lambda_2 = 1 \to z=0,\ x+y=1,\ y=1-x$; $x^2+(1-x)^2=1$; $\therefore (0,1,0),\ f=1$; $(1,0,0),\ f=1$

$\lambda_2 \neq 1 \to x = y = \frac{z}{1-\lambda_2}$; $x^2+x^2=1$; $x = \pm 1/\sqrt{2} = y$

$x+x+z=1$
$z = 1-2x = 1 \mp \sqrt{2}$

$\therefore \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2}\right)$

$\hookrightarrow f = 4-\sqrt{2}$ $\qquad \hookrightarrow f = 4+\sqrt{2}$

$\therefore$ Since f is continuous and the curve of intersection is compact, mins $(0,1,0)$, $(1,0,0)$ and max (farthest) is $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2}\right)$.

Exam '10 - Q7

Find the Extreme Values of $f(x,y,z)=x$ in the intersection of the unit sphere $x^2+y^2+z^2=1$ and the plane $x+y+z=1$. Justify.

$h(x,y,z,\lambda_1,\lambda_2)=x-\lambda_1(x^2+y^2+z^2-1)-\lambda_2(x+y+z-1)$

$$\begin{cases} h_x = 1-2\lambda_1 x-\lambda_2=0 \\ h_y=-2\lambda_1 y-\lambda_2=0 \\ h_z=-2\lambda_1 z-\lambda_2=0 \\ h_{\lambda_1}=-(x^2+y^2+z^2-1)=0 \\ h_{\lambda_2}=-(x+y+z-1)=0 \end{cases}$$

$-2\lambda_1 y=-2\lambda_1 z$

$0=2\lambda_1(y-z)$

$\rightarrow \lambda_1=0$ or $y=z$

if $\lambda_1=0$

$\lambda_2=0$, $1=0$ ... what? $\therefore \lambda_1\neq 0$

if $y=z$

$x+y+y=1$

$x=1-2y$

$(1-2y)^2+y^2+y^2=1$

$6y^2-4y=0$

$y(6y-4)=0$

$\rightarrow y=0$ or $y=\frac{2}{3}$

$y=0,\ x=1 \rightarrow (1,0,0) \rightarrow f=1$

$y=\frac{2}{3},\ x=-\frac{1}{3} \rightarrow (-\frac{1}{3},\frac{2}{3},\frac{2}{3}) \rightarrow f=-\frac{1}{3}$

$\therefore$ Since f is continuous and the curve of intersection is compact, therefore the extrema will exist by EVT

$(1,0,0)$ is max and $(-\frac{1}{3},\frac{2}{3},\frac{2}{3})$ is min.

Exam '12 - Q9

Find the maximum value of $f(x,y,z)=xz+yz$ on the solid ellipsoid $x^2+2y^2+6z^2\leq 12$.

Since f is continuous and ellipsoid is compact, f will attain a global max and min on the ellipsoid by EVT.
The extrema may occur either on the interior on on the boundary of the ellipsoid.

Interior

$\nabla f=(z,z,x+y)$

$\begin{cases} z=0 \\ y=-x \end{cases}$ CP is $(x,-x,0)$ $\rightarrow f(x,-x,0)=0$, cannot be max

Boundary

$h(x,y,z,\lambda)=xz+yz-\lambda(x^2+2y^2+6z^2-12)$

$h_x=z-2x\lambda=0$

$h_y=z-4y\lambda=0$

$h_z=x+y-12z\lambda=0$

$h_\lambda=-(x^2+2y^2+6z^2-12)=0$

(Ignore $z=0$) cannot be max

$\rightarrow 2x=4y$

$x=2y$

$\rightarrow y=\pm z$

$\rightarrow y=\pm 1$

$\therefore$ CPs $=(2,1,1);(2,1,-1),(-2,-1,1),(-2,-1,-1)$

$f(2,1,1)=3$

$f(2,1,-1)=-3$

$f(-2,-1,1)=-3$

$f(-2,-1,-1)=3$

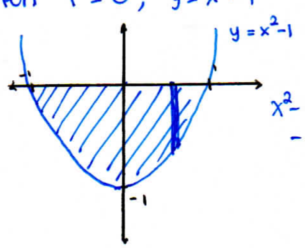
$\therefore$ Global max $(2,1,1)$, $(-2,-1,-1)$.

Exam '16 - Q9b.

Evaluate $\int_D \|\nabla f\|^2 dA$, where $f(x,y)=y-x^2+1$ and $D=\{(x,y)\mid f(x,y)\geq 0, y\leq 0\}$.

$\nabla f(x,y)=(-2x,1)$

when $f=0$, $y=x^2-1$



$x^2-1\leq y\leq 0$

$-1\leq x\leq 1$

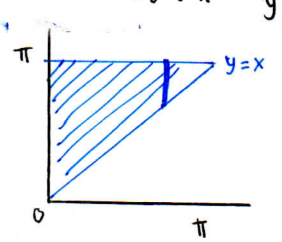
$$=\int_{-1}^{1}\int_{x^2-1}^{0} 4x^2+1\,dy\,dx = \int_{-1}^{1} 4x^2y+y\Big|_{x^2-1}^{0}dx$$

$$=\int_{-1}^{1}-(4x^2+1)(x^2-1)\,dx = \left[-\frac{4x^5}{5}+\frac{4x^3}{3}-\frac{x^3}{3}+x\right]_{-1}^{1}=\frac{12}{5}$$

$$=\int_{-1}^{1}-4x^4+4x^2-x^2+1\,dx$$

Exam '10 - Q9a.

Evaluate $\int_0^\pi\int_x^\pi \frac{\sin y}{y}\,dy\,dx$.

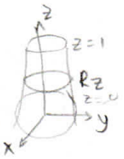


$$=\int_0^\pi\int_0^y \frac{\sin y}{y}\,dx\,dy$$

$$=\int_0^\pi \frac{x\sin y}{y}\Big|_0^y dy = \int_0^\pi \sin y\,dy = -\cos y\Big|_0^\pi = 2$$

Exam '15 - Q12

Evaluate $\int_B z\,dV$, where B is the region bounded by the planes $z=0$, $z=1$ and the surface $(z+1)\sqrt{x^2+y^2}=1$



$$\int_0^1 \left(\iint_{R_z} z\,dx\,dy\right)dz = \int_0^1 z\,(\text{area of } R_z)\,dz$$
$$= \int_0^1 z\left(\pi\left(\frac{1}{1+z}\right)^2\right)dz = \pi\int_0^1\left(\frac{1}{1+z} - \frac{1}{(1+z)^2}\right)dz$$
$$= \pi\left[\ln|1+z| + \frac{1}{1+z}\right]_0^1 = \pi(\ln 2 - 1/2) \;//$$

Final '09 - Q11

Evaluate $\int_B e^{x+y+z}dV$ where B is the region in $\mathbb{R}^3$ bounded by the planes $y=1$, $y=-x$, $z=-x$ and the coordinate planes, $x=0$ and $z=0$.

Fix x, $-1 \le x \le 0$

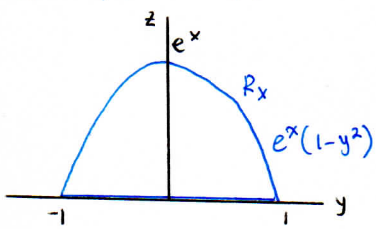
$$\begin{cases} 0 \le z \le -x \\ -x \le y \le 1 \end{cases}$$

$$= \int_{-1}^0 \left(\iint e^{x+y+z}\,dA\right)dx = \int_{-1}^0\int_{-x}^1\int_0^{-x} e^{x+y+z}\,dz\,dy\,dx = \int_{-1}^0\int_{-x}^1 e^y - e^{x+y}\,dy\,dx$$
$$= \int_{-1}^0 e - e^{x+1} + 1 - e^{-x}\,dx = 3 - e \;//$$

Final '16 - Q11

A train track runs along the x-axis. A curved roof is stretched over the track. The roof touches the ground (xy-plane) in straight lines $y=1$, $y=-1$. For a given value a of x, the intersection of the roof with the plane $x=a$ is a parabola with the highest point directly over the train track at height e^a. Find the volume enclosed by the roof and the x-y plane between $x=0$ and $x=1$

Fix x for $0 \le x \le 1$.



$$V = \iiint_B 1\,dV = \int_0^1 \iint_{R_x} 1\,dA\,dx$$

R_x is a parabolic region in the plane $x=a$. The equation of the bounding parabola must be $z = ky^2 + l$, k, l are constants.

$\rightarrow (0, e^x)$ $\quad e^x = l$

$\rightarrow (1, 0)$ $\quad k = -l$

$\therefore\ z = e^x - e^x y^2$.

$$R_x \begin{cases} -1 \le y \le 1 \\ 0 \le z \le e^x(1-y^2) \end{cases}$$

$$= \int_0^1\int_{-1}^1\int_0^{e^x(1-y^2)} 1\,dz\,dy\,dx$$
$$= \int_0^1\int_{-1}^1 e^x(1-y^2)\,dy\,dx = \int_0^1 \left[e^x\left(y - \frac{y^3}{3}\right)\right]_{-1}^1 dx = \int_0^1 \frac{4e^x}{3}\,dx = \frac{4}{3}(e-1) \;//$$