MATB41 Final Review Sem.

(Qs from Fall 2017)

Q4. Let $f(x,y) = e^{xy}\sin(x+y)$

(a) In what direction(s), starting at $(0, \pi/2)$, is f increasing the fastest?

(b) In what direction(s), starting at $(0, \pi/2)$, is f changing at 50% of its maximum rate?

(a) Get the gradient of f: $\nabla f = (ye^{xy}\sin(x+y) + \cos(x+y)e^{xy},\ xe^{xy}\sin(x+y) + \cos(x+y)e^{xy})$

$\nabla f(0, \pi/2) = (\pi/2, 0)$.

f is increasing fastest in direction $(\pi/2, 0)$.

{ Because the gradient direction is the direction of maximal increase.

(b) Rate of change in direction u is the directional derivative $D_u f(x) = \nabla f(x) \cdot \frac{u}{\|u\|}$.

Rate of the fastest increase in the given f is $\|\nabla f(0, \pi/2)\| = \|(\pi/2, 0)\| = \pi/2$

$\nabla f(0, \pi/2) \cdot u = \frac{\pi}{2} \cdot 0.50 = \frac{\pi}{4}$

(increase 50%.)

* u is unit vector in the required direction.

$= \frac{\pi}{2}(1)\cos\theta$, θ is the angle between u and $(\pi/2, 0)$

$\Rightarrow \frac{\pi}{2}\cos\theta = \frac{\pi}{4}$. When u makes an angle of $\pm \pi/3$ with +ve x-axis.

$\cos\theta = \frac{1}{2}$, $\theta = \pm\frac{\pi}{3}$ $\therefore u = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Q5. Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x,y,z) = (x^2y, yz^2)$ and let $g: \mathbb{R}^2 \to \mathbb{R}^5$ be given by $g(x,y) = (xy, 2x^2, x+y, -x, y)$. Find Df and Dg and find $D(g \circ f)$ using the Chain Rule.

$$Df = \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & z^2 & 2yz \end{pmatrix} \qquad Dg = \begin{pmatrix} y & x \\ 4x & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Dg(f) = Dg(x^2y, yz^2) = \begin{pmatrix} yz^2 & x^2y \\ 4x^2y & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D(g \circ f) = Dg(f)Df = \begin{pmatrix} yz^2 & x^2y \\ 4x^2y & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} 2xy & x^2 & 0 \\ 0 & z^2 & 2yz \end{bmatrix} = \begin{pmatrix} 2xy^2z^2 & 2x^2yz^2 & 2x^2y^2z \\ 8x^3y^2 & 4x^4y & 0 \\ 2xy & x^2+z^2 & 2yz \\ -2xy & -x^2 & 0 \\ 0 & z^2 & 2yz \end{pmatrix}$$

Q6. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 1-(x^2+y^2-1)^2$

(a) Characterize and sketch several level curves of f being careful to indicate where f is zero, positive, negative and not defined. What is the range of f?

(b) Find the CPs of f and determine the local and global extreme of f or explain why such extrema do not exist.

(c) Find the equation of the tangent plane to the graph of f at $(1,1,f(1,1))$?

(a) $1-(x^2+y^2-1)^2 = c.$

$x^2+y^2 = 1 \pm \sqrt{1-c}, \quad c \le 1$

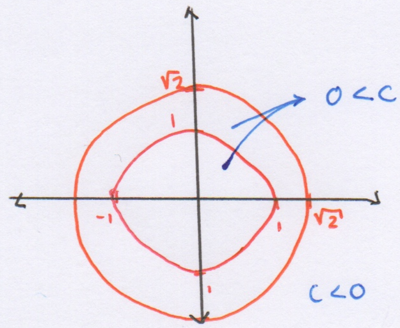
$c=1$ $\quad x^2+y^2=1$

$0<c\leq 1$ $\quad$ Circles centered at $(0,0)$ with radius $\sqrt{1\pm\sqrt{1-c}}$.

$c=0$ $\quad$ point $(0,0)$

$x^2+y^2=2$

$c<0$ $\quad$ Circles centered at $(0,0)$ with radius $\sqrt{1+\sqrt{1-c}}$



(b) $f_x = -2(x^2+y^2-1)(2x) = -4x(x^2+y^2-1) = 0, \quad x=0$ or $x^2+y^2=1$

$f_y = -2(x^2+y^2-1)(2y) = -4y(x^2+y^2-1) = 0, \quad y=0$ or $x^2+y^2=1$

All points on the circle $x^2+y^2=1$ and the point $(0,0)$ are critical.

From (a), there is a local and global maximum of 1 on the circle $x^2+y^2=1$ and a local minimum of 0 at $(0,0)$.

There is no global minimum as $f(x,y) \to -\infty$ as $x,y \to \infty$

(c) The tangent plane is $z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 0 - 4(x-1) - 4(y-1)$

$= -4x+4-4y+4$

$= -4x-4y+8$ or $4x+4y+z=8$

Tangent plane eq. $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Q7. Let $f(x,y)=2x^4-xy^2+2y^2$. Find all the critical points of f. For each critical point, determine if that point is a local min, max or saddle.

$f(x,y)$ is polynomial so it is differentiable for all $(x,y)\in\mathbb{R}^2$. (CPs when $\nabla f=0$)

$f_x=8x^3-y^2=0$ $\qquad$ $f_y=-2xy+4y=0$

$\qquad\qquad\qquad\qquad\quad =2y(-x+2)$

$\underline{y=0}$, $x=0$ $\qquad\qquad\qquad y=0$ or $x=2$.

$\underline{x=2}$, $y^2=64$

$\qquad y=\pm 8.$ $\qquad\qquad \therefore$ CPs $(0,0),(2,8),(2,-8)$

The Hessian matrix is $Hf=\begin{pmatrix}24x^2 & -2\\ -2y & -2x+4\end{pmatrix}$

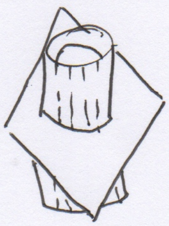
$Hf(0,0)=\begin{pmatrix}0&0\\0&4\end{pmatrix}$, det $Hf(0,0)=0$ so this is degenerate case.

Since $f(x,y)\geq 0$ for (x,y) near $(0,0)$, there is a local minimum at $(0,0)$.

$Hf(2,8)=\begin{pmatrix}96&-16\\-16&0\end{pmatrix}$, det $Hf(2,8)=-16\times16<0$ $\therefore$ saddle

$Hf(2,-8)=\begin{pmatrix}96&16\\16&0\end{pmatrix}$, det $Hf(2,-8)=-16\times16<0$ $\therefore$ saddle.

Q8. Let $f(x,y,z)=x+2y+3z$. Find the global extrema of f on the intersection of the surfaces $x^2+y^2=1$ and $x-y+z=1$.



Since f is a polynomial, f is cont on $\mathbb{R}^3$. The curve of intersection is an ellipse, it is compact in $\mathbb{R}^3$. Then, the EVT ensures f will attain both global max and min on ellipse.

Constraints $g_1(x,y,z)=x^2+y^2-1$, $g_2(x,y,z)=x-y+z-1$

$h(x,y,z,\lambda,\mu)=x+2y+3z-\lambda(x^2+y^2-1)-\mu(x-y+z-1)$

$h_x=1-2x\lambda-\mu=0$

$h_y=2-2y\lambda+\mu=0$ $\qquad \therefore \left(\mp\frac{2}{\sqrt{29}},\pm\frac{5}{\sqrt{29}},1\pm\frac{7}{\sqrt{29}}\right)$

$h_z=3-\mu=0$

$h_\lambda=-(x^2+y^2-1)=0$ $\qquad f(-,+,+)=3+\sqrt{29}$ (max)

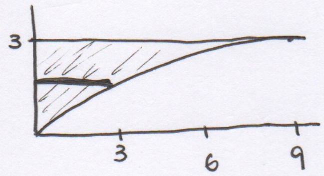
$h_\mu=-(x-y+z-1)=0$ $\qquad f(+,-,-)=3-\sqrt{29}$ (min)

Q9. (a) Compute $\iint_D (1+2y\cos x)\,dA$, where D is the region bounded by the curve $y=\sqrt{x}$ and the lines $x=0$ and $y=3$.

(b) Rewrite the integral $\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} f(x,y)\,dy\,dx$ with the order of integration reversed.

(c) Give an integral in polar coordinates (r,θ) which is equivalent to $\int_0^4\int_3^{\sqrt{25-x^2}} dy\,dx$

(a)

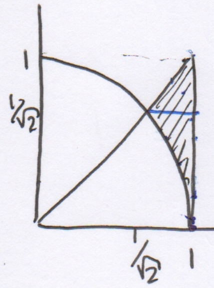


$$\iint_D (1+2y\cos x)\,dA = \int_0^3\int_0^{y^2} 1+2y\cos x\,dx\,dy$$

$$= \int_0^3 \left[x + 2y\sin x\right]_0^{y^2} dy = \int_0^3 y^2 + 2y\sin y^2\,dy$$

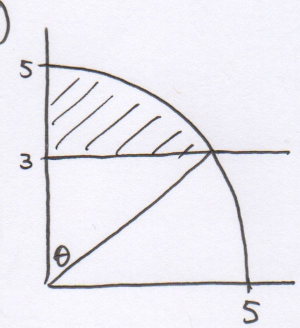
$$= \left[\frac{y^3}{3} - \cos y^2\right]_0^3 = 9-\cos 9+1 = 10-\cos 9.$$

(b)



$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} f(x,y)\,dy\,dx = \int_0^{1/\sqrt{2}}\int_{\sqrt{1-y^2}}^{1} f(x,y)\,dx\,dy + \int_{1/\sqrt{2}}^{1}\int_y^1 f(x,y)\,dx\,dy$$

(c)



$y = r\sin\theta = 3$

$r = 3\csc\theta$

circle $r=5$.

$\frac{5}{3} = \csc\theta$, $\theta = \operatorname{arccsc}\left(\frac{5}{3}\right)$.

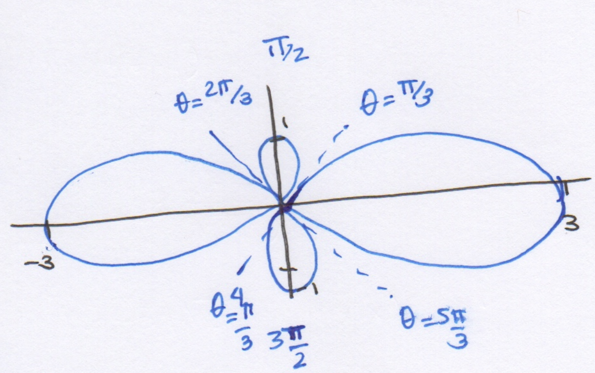
$$\int_0^4\int_3^{\sqrt{25-x^2}} dy\,dx = \int_{\operatorname{arccsc}(\frac{5}{3})}^{\pi/2}\int_{3\csc\theta}^{5} r\,dr\,d\theta$$

Q10. Sketch the curve given by the polar equation $r = 1+2\cos(2\theta)$.

$r=0 \rightarrow 1+2\cos(2\theta)=0$

$\cos(2\theta) = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$\therefore$ graph will be tangent to $\theta=\frac{\pi}{3}$, $\theta=\frac{2\pi}{3}$, $\theta=\frac{4\pi}{3}$, $\theta=\frac{5\pi}{3}$.



θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π
r	3	0	-1	0	3	0	-1	0	3

Q11. Find, in terms of a, the volume of the first octant region bounded ~~by~~ above by the plane $z = x+y$ and bounded on one side by the ~~cy~~ cylinder $x^2+y^2=a^2$, where $a>0$.

$x = \rho\cos\phi$
$y = \rho\sin\phi$

$0 \le z \le x+y$.

Use cylindrical coordinates.

$$\int_B 1\, dV = \int_0^{\pi/2}\int_0^a\int_0^{r(\cos\theta+\sin\theta)} r\, dz\, dr\, d\theta = \int_0^{\pi/2}(\cos\theta+\sin\theta)\int_0^a r^2\, dr\, d\theta$$

$$= \int_0^{\pi/2}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^a d\theta = \frac{a^3}{3}\left[\sin\theta-\cos\theta\right]_0^{\pi/2} = \frac{2a^3}{3}$$

Q12. Let B be the interior of the unit sphere, $x^2+y^2+z^2=1$.

(a) Evaluate $\int_B (x^2+y^2+z^2)\, dV$.

(b) Explain why this should or should not give the same answer as $\int_B 1\, dV$.

(a) Using spherical coordinates:

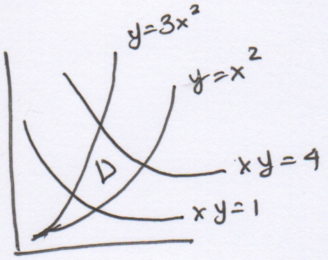
$$\int_B (x^2+y^2+z^2)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 |-\rho^2\sin\phi|\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^4\sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\left[\frac{\rho^5}{5}\right]_0^1 \sin\phi\, d\theta\, d\phi$$

$$= \frac{1}{5}(2\pi)\int_0^{\pi}\sin\phi\, d\phi = \frac{2\pi}{5}\left[-\cos\phi\right]_0^{\pi} = \frac{4\pi}{5}.$$

(b) Not the same since $x^2+y^2+z^2=1$ only on the surface not the interior.

Q13. Use a change of variable to evaluate $\iint_D xy\, dA$, where D is the first quadrant region bounded by $xy=1$, $xy=4$, $y=x^2$ and $y=3x^2$.



$u = \frac{y}{x^3}$, $v = xy$ then $1 \le u \le 3$, $1 \le v \le 4$.

$$Jh(x) = \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} -2y/x^3 & 1/x^2 \\ y & x \end{pmatrix} = -\frac{3y}{x^2}.$$

$$Jg(u) = \frac{\partial(x,y)}{\partial(u,v)} = -\frac{x^2}{3y} = -\frac{1}{3u}.$$

Therefore, $\int_D xy\, dA = \int_{D^*} v\left|-\frac{1}{3u}\right| dA = \frac{1}{3}\int_1^3\int_1^4 \frac{v}{u}\, dv\, du = \frac{1}{3}\int_1^3 \frac{1}{u}\left[\frac{v^2}{2}\right]_1^4 du$

$$= \frac{1}{4}\int_1^3 \frac{1}{4}(15)\, du = \frac{15}{6}\left[\ln u\right]_1^3 = \frac{5}{2}\ln 3.$$

Other q's.

Evaluate $\iiint_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$, where S is the two spheres $x^2+y^2+z^2=a^2$ and

✳

$x^2+y^2+z^2=b^2$, where $0<a<b$.

$x=\rho\sin\phi\cos\theta$ $\quad 0\le\rho<\infty$

$y=\rho\sin\phi\sin\theta$ $\quad 0\le\theta\le 2\pi$

$z=\rho\cos\phi$ $\quad 0\le\phi\le\pi$.

Use spherical polar coordinates: (ρ, θ, ϕ).

For the two spheres, $\rho=a$ and $\rho=b$., $a\le\rho\le b$, R_ρ is the rectangle $\begin{cases} 0\le\theta\le 2\pi \\ 0\le\phi\le\pi \end{cases}$

$$\int_S \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}} = \int_a^b\int_0^{2\pi}\int_0^{\pi} \frac{\rho^2\sin\phi}{\rho^3}\,d\phi\,d\theta\,d\rho. = \int_a^b \frac{1}{\rho}\int_0^{2\pi}\left(-\cos\phi\,\Big|_0^{\pi}\right)d\theta\,d\rho$$

$$= \int_a^b \frac{1}{\rho}\int_0^{2\pi} 2\,d\theta\,d\rho = 4\pi\int_a^b \frac{1}{\rho}\,d\rho = 4\pi\ln\rho\,\Big|_a^b = 4\pi\ln\left(\frac{b}{a}\right)$$

Evaluate $\int_0^1\int_{e^y}^{e} \frac{x}{\ln x}\,dx\,dy$

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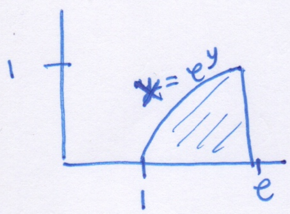
$e^y\le x\le e$, $0\le y\le 1$

$1\le x\le e$ $\quad 0\le y\le\ln x$



$$\int_0^1\int_{e^y}^{e} \frac{x}{\ln x}\,dx\,dy = \int_1^e\int_0^{\ln x} \frac{x}{\ln x}\,dy\,dx = \int_1^e \left[y\,\frac{x}{\ln x}\right]_0^{\ln x} dx$$

$$= \int_1^e x\,dx = \left[\frac{x^2}{2}\right]_1^e = \frac{1}{2}(e^2-1)$$