Q4. Let \( f(x, y) = e^{xy} \sin(xy) \)
(a) In what direction(s), starting at \((0, \pi/2)\), is \( f \) increasing the fastest?
(b) In what direction(s), starting at \((0, \pi/2)\), is \( f \) changing at 50% of its maximum rate?

(a) Get the gradient of \( f \): \( \nabla f = (ye^{xy} \sin(xy) + \cos(xy)e^{xy}, xe^{xy} \sin(xy) + \cos(xy)e^{xy}) \)

\( \nabla f(0, \pi/2) = (\pi/2, 0) \). Because the gradient direction is the direction of maximal increase.

(b) Rate of change in direction \( u \) is the directional derivative \( D_u f(x) = \nabla f(x) \cdot u \).

Rate of the fastest increase in the given \( f \) is \( \|\nabla f(0, \pi/2)\| = \| (\pi/2, 0) \| = \pi/2 \)

\( \nabla f(0, \pi/2) \cdot u = \frac{\pi}{2} \cdot 0.50 = \frac{\pi}{4} \)

* \( u \) is unit vector in the required direction.

* When \( u \) makes an angle of \( \pm \frac{\pi}{3} \) with the \( x \)-axis.

\( \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3} \Rightarrow u = \left( \frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right) \)

Q5. Let \( f: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be given by \( f(x, y, z) = (x^2y, y^2z) \) and let \( g: \mathbb{R}^2 \rightarrow \mathbb{R}^5 \) be given by \( g(x, y) = (xy, 2x^2, x+y, -x, y) \). Find \( Df \) and \( Dg \) and find \( D(g \circ f) \) using the Chain Rule.

\( Df = \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2z & 2yz \end{pmatrix} \quad Dg = \begin{pmatrix} y & x \\ 4x & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( Dg(f) = Dg(x^2y, y^2z) = \begin{pmatrix} y^2 & x^2y & 0 \\ 4xy & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \)

\( D(g \circ f) = Dg(f) Df = \begin{pmatrix} y^2 & x^2y & 0 \\ 4xy & 0 & 2yz \end{pmatrix} \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2z & 2yz \end{pmatrix} = \begin{pmatrix} 2x^2y^2x & 2x^2y^2 & 2x^2yz \\ 8x^3y^2 & 4x^4y & 0 \\ 2xy & x^2z & 2yz \\ -2xy & -x^2 & 0 \\ 0 & 2z & 2yz \end{pmatrix} \)
Q.6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be given by \( f(x, y) = 1 - (x^2 + y^2)^2 \)

(a) Characterize and sketch several level curves of \( f \) being careful to indicate where \( f \) is zero, positive, negative and not defined. What is the range of \( f \)?

(b) Find the \( CPs \) of \( f \) and determine the local and global extreme of \( f \) or explain why such extrema do not exist.

(c) Find the equation of the tangent plane to the graph of \( f \) at \((1,1,f(1,1))\)?

\[
\begin{align*}
\text{if } & \quad 0 < C < 1, \quad \text{cicles centered at } (0,0) \text{ with radius } \sqrt{1 - \sqrt{1 - C}}, \\
\text{if } & \quad C = 0, \quad \text{point } (0,0), \quad x^2 + y^2 = 1, \\
\text{if } & \quad C < 0, \quad \text{cicles centered at } (0,0) \text{ with radius } \sqrt{1 - \sqrt{1 - C}}.
\end{align*}
\]

(b) 
\[
\begin{align*}
f_x &= -2(x^2 + y^2 - 1)(2x) = -4x(x^2 + y^2 - 1) = 0, \quad x = 0 \quad \text{or} \quad x^2 + y^2 = 1, \\
f_y &= -2(x^2 + y^2 - 1)(2y) = -4y(x^2 + y^2 - 1) = 0, \quad y = 0 \quad \text{or} \quad x^2 + y^2 = 1.
\end{align*}
\]

All points on the circle \( x^2 + y^2 = 1 \) and the point \((0,0)\) are critical.

From (a), there is a local and global maximum of 1 on the circle \( x^2 + y^2 = 1 \) and a local minimum of 0 at \((0,0)\).

There is no global minimum as \( f(x,y) \to -\infty \) as \( x,y \to \infty \).

(c) The tangent plane is 
\[
z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)
\]
\[
= 0 - 4(x-1) - 4(y-1)
\]
\[
= -4x + 4 - 4y + 4
\]
\[
= -4x - 4y + 8 \quad \text{or} \quad 4x + 4y + z = 8.
\]

Tangent plane eq. \( z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \).
Q7. Let \( f(x,y) = 2x^4 - xy^2 + 2y^2 \). Find all the critical points of \( f \). For each critical point, determine if that point is a local min, max or saddle.

\( f(x,y) \) is polynomial so it is differentiable for all \( (x,y) \in \mathbb{R}^2 \). (CPs when \( \nabla f = 0 \))

\[
\begin{align*}
    f_x &= 8x^3 - y^2 = 0 \\
    f_y &= -2xy + 4y = 0 \\
    y &= 0 \text{ or } x = 2 \\
    x &= 2, \quad y^2 = 64 \\
    y &= \pm 8.
\end{align*}
\]

\[\therefore \text{CPs } (0,0), (2,8), (2,-8)\]

The Hessian matrix is \( H_f = \begin{pmatrix} 24x^2 & -2 \\ -2y & -2x + 4 \end{pmatrix} \)

\[H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \quad \det H_f(0,0) = 0 \text{ so this is degenerate case.} \]

\[\text{Since } f(x,y) > 0 \text{ for } (x,y) \text{ near } (0,0), \text{ there is a local minimum at } (0,0).\]

\[H_f(2,8) = \begin{pmatrix} 96 & -16 \\ -16 & 0 \end{pmatrix}, \quad \det H_f(2,8) = -16 \times 16 < 0 \therefore \text{saddle}\]

\[H_f(2,-8) = \begin{pmatrix} 96 & 16 \\ 16 & 0 \end{pmatrix}, \quad \det H_f(2,-8) = -16 \times 16 < 0 \therefore \text{saddle}.\]

Q8. Let \( f(x,y,z) = x + 2y + 3z \). Find the global extrema of \( f \) on the intersection of the surfaces \( x^2 + y^2 = 1 \) and \( x + y + z = 1 \).

Since \( f \) is a polynomial, \( f \) is continuous on \( \mathbb{R}^3 \). The curve of intersection is an ellipse, it is compact in \( \mathbb{R}^3 \). Then, the EVT ensures \( f \) will attain both global max and min on ellipse.

Constraints \( g_1(x,y,z) = x^2 + y^2 - 1 \), \( g_2(x,y,z) = x + y + z - 1 \)

\[\begin{align*}
    h(x,y,z,\lambda,\mu) &= x + 2y + 3z - 2(x^2 + y^2 - 1) - \lambda(x+y+z-1) \\
    h_x &= 1 - 2x\lambda - y = 0 \\
    h_y &= 2 - 2y\lambda + 4y = 0 \\
    h_z &= 3 - \lambda = 0 \\
    h_\lambda &= -(x^2 + y^2 - 1) = 0 \\
    h_\mu &= -(x + y + z - 1) = 0
\end{align*}\]

\[\therefore (\pm \frac{2}{\sqrt{29}}, \pm \frac{5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}})\]

\[f(-, +, +) = 3 + \sqrt{29} \quad \text{(max)} \]

\[f(+, -, -) = 3 - \sqrt{29} \quad \text{(min)}\]
Q9. (a) Compute \( \iint_D (1 + 2y \cos x) \, dA \), where \( D \) is the region bounded by the curve \( y = \sqrt{x} \) and the lines \( x = 0 \) and \( y = 3 \).

(b) Rewrite the integral \( \int_0^1 \int_0^x f(x, y) \, dy \, dx \) with the order of integration reversed.

(c) Give an integral in polar coordinates \((r, \theta)\) which is equivalent to \( \int_0^4 \int_3 \sqrt{25-x^2} \, dy \, dx \)

\[
\iint_D (1 + 2y \cos x) \, dA = \int_0^3 \int_0^x (1 + 2y \cos x) \, dx \, dy \\
= \int_0^3 \left[ x + 2y \sin x \right]_0^x \, dy = \int_0^3 y^2 + 2y \sin y^2 \, dy \\
= \left[ \frac{y^3}{3} - \cos y^2 \right]_0^3 = 9 \cos 9 + 1 = 10 - \cos 9.
\]

\[
\int_0^1 \int_0^x f(x, y) \, dy \, dx = \int_0^{\sqrt{25}} \int_0^1 f(x, y) \, dx \, dy + \int_0^3 \int_{\sqrt{25}}^{1} f(x, y) \, dx \, dy
\]

\[
\int_0^4 \int_3 \sqrt{25-x^2} \, dy \, dx = \int_0^{\pi/2} \int_0^{\arccsc(\frac{5}{3})} r \, dr \, d\theta
\]

Q10. Sketch the curve given by the polar equation \( r = 1 + 2 \cos(2\theta) \).

\[ r = 0 \rightarrow 1 + 2 \cos(2\theta) = 0 \]

\[ \cos(2\theta) = -\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \]

\[ \therefore \text{graph will be tangent to } \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3} \]

\[ \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3} \]

\[ \begin{array}{c|cc|cc|cc|cc|cc}
\theta & 0 & \pi/6 & \pi/3 & 2\pi/3 & \pi & 4\pi/3 & 3\pi/2 & 5\pi/3 & 2\pi \\
\theta & 3 & 0 & -1 & 0 & 3 & 0 & -1 & 0 & 3
\end{array} \]
Q11. Find, in terms of \( a \), the volume of the first octant region bounded above by the plane \( z = x + y \) and bounded on one side by the cylinder \( x^2 + y^2 = a^2 \) where \( a > 0 \).

Use cylindrical coordinates.

\[
\int_B \, dV = \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = \int_0^{\pi/2} (\sin \theta) \, d\theta \int_0^a \frac{r^3}{3} \, dr = \frac{a^3}{3} \left[ \sin \theta - \frac{\theta}{2} \right]_0^{\pi/2} = \frac{2a^3}{3}
\]

Q12. Let \( B \) be the interior of the unit sphere, \( x^2 + y^2 + z^2 = 1 \).

(a) Evaluate \( \int_B (x^2 + y^2 + z^2) \, dV \).

(b) Explain why this should or should not give the same answer as \( \int_B 1 \, dV \).

(a) Using spherical coordinates:

\[
\int_B (x^2 + y^2 + z^2) \, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{4\pi}{3}
\]

(b) Not the same since \( x^2 + y^2 + z^2 = 1 \) only on the surface, not the interior.

Q13. Use a change of variable to evaluate \( \iint_D xy \, dA \), where \( D \) is the first quadrant region bounded by \( xy = 1 \), \( xy = 4 \), \( y = x^2 \) and \( y = 3x^2 \).

\[
u = \frac{y}{x^3}, \quad v = xy. \quad \text{then} \quad 1 \leq u \leq 3, \quad 1 \leq v \leq 4.
\]

\[
\iint_D xy \, dA = \int_0^1 \int_0^4 \frac{v}{u^2} \, dv \, du = \frac{1}{3} \int_0^3 \left( \frac{v}{u} \right)^4 \, du = \frac{1}{3} \int_0^3 \frac{15}{u} \, du = \frac{15}{6} \ln u \bigg|_1^3 = \frac{5}{2} \ln 3.
\]
Evaluate \( \iint_S \frac{dx \, dy \, dz}{(x^2+y^2+z^2)^{3/2}} \), where \( S \) is the two spheres \( x^2+y^2+z^2 = a^2 \) and \( x^2+y^2+z^2 = b^2 \), where \( 0 < a < b \).

Use spherical polar coordinates: \((p, \theta, \phi)\).

For the two spheres, \( p = a \) and \( p = b \), \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \phi \leq \pi \).

\[
\int_a^b \int_0^{2\pi} \int_0^\pi \frac{p^2 \sin \phi}{p^3} \, d\phi \, d\theta \, dp = \int_a^b \left[ -\frac{1}{p^2} \cos \phi \right]_0^\pi \, d\theta \, dp
\]

\[
= \int_a^b \frac{1}{p} \int_0^{2\pi} \cos \phi \, d\phi \, dp = 4\pi \int_a^b \frac{1}{p} \, dp = 4\pi \ln \frac{b}{a}
\]

Evaluate \( \int_0^1 \int_0^e \frac{x}{\ln x} \, dx \, dy \)

\( e^y \leq x \leq e, \ 0 \leq y \leq 1 \)

\( 1 \leq x \leq e, \ 0 \leq y \leq \ln x \)

\[
\int_0^1 \int_0^e \frac{x}{\ln x} \, dx \, dy = \int_0^e \int_0^1 \frac{x}{\ln x} \, dy \, dx = \int_0^e \left[ \frac{x}{\ln x} \right]_0^1 \, dx
\]

\[
= \int_0^e x \, dx = \left[ \frac{x^2}{2} \right]_1^e = \frac{1}{2} (e^2 - 1)
\]