MATB41 - Final Review Seminar

Taylor Series

(x,y) around $(0,0)$:

$$F(x,y) = F(0,0) + x\frac{\partial F}{\partial x}(0,0) + y\frac{\partial F}{\partial y}(0,0)$$
$$+ \frac{1}{2!}\left[x^2\frac{\partial^2 F}{\partial x^2}(0,0) + 2xy\frac{\partial^2 F}{\partial x \partial y}(0,0) + y^2\frac{\partial^2 F}{\partial y^2}(0,0)\right] + \cdots$$

else:

$$F(x,y) = F(a,b) + \frac{\partial F}{\partial x}(x-a) + \frac{\partial F}{\partial y}(y-b)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 F}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 F}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 F}{\partial y^2}(y-b)^2\right] + \cdots$$

★ Common series around $a=0$ are:

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}, \quad |x|<\infty \qquad\qquad \frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k, \quad |x|<1$$

$$\cos x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}, \quad |x|<\infty \qquad\qquad \ln(1+x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1}, \quad |x|<1$$

$$\sin x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}, \quad |x|<\infty \qquad\qquad \arctan x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}, \quad |x|<1$$

2015 Final Q5(a)
Compute the 4th degree Taylor polynomial about the origin of $f(x,y) = e^{y^2}\sin(x+y)$

$$e^{y^2} = \sum_{k=0}^{\infty}\frac{(y^2)^k}{k!} = \frac{(y^2)^0}{0!} + \frac{(y^2)^1}{1!} + \frac{(y^2)^2}{2!} + \frac{(y^2)^3}{3!} + \cdots \quad, |y^2|<\infty$$
$$= 1 + y^2 + \frac{y^4}{2} + \frac{y^6}{6} + \cdots$$

$$\sin(x+y) = \sum_{k=0}^{\infty}(-1)^k\frac{(x+y)^{2k+1}}{(2k+1)!} = (-1)^0\frac{(x+y)^1}{1!} + (-1)^1\frac{(x+y)^3}{3!} + (-1)^2\frac{(x+y)^5}{5!} + \cdots \quad, |x+y|<\infty$$
$$= (x+y) - \frac{(x+y)^3}{6} + \frac{(x+y)^5}{120} + \cdots$$

$$T = e^{y^2}\sin(x+y) = \left(1 + y^2 + \frac{y^4}{2} + \frac{y^6}{6} + \cdots\right)\left((x+y) - \frac{(x+y)^3}{6} + \frac{(x+y)^5}{120} + \cdots\right)$$

$$= (x+y) - \frac{(x+y)^3}{6} + y^2(x+y) - y^2\frac{(x+y)^3}{6} + \frac{y^4}{2}(x+y) - \frac{y^4}{2}\frac{(x+y)^3}{6} + \cdots$$

this should be enough, I am just extra.

$$T_4 = \underbrace{(x+y) - \frac{(x+y)^3}{6} + y^2(x+y)} = (x+y)(1+y^2) - \frac{(x+y)^3}{6}$$

$$= (x+y)\left[1+y^2 - \frac{(x+y)^2}{6}\right] = (x+y)\left[1+y^2 - \frac{x^2}{6} - \frac{2xy}{6} - \frac{y^2}{6}\right]$$

$$= (x+y)\left(1 + \frac{5y^2}{6} - \frac{xy}{3} - \frac{x^2}{6}\right)$$

Hessian Form

* If H.F. is positive definite, then the Critical Point is a local minimum
* If H.F. is negative definite, then the Critical Point is a local maximum
* Otherwise, the Critical Point is a saddle point.

Let A be $n \times n$ and symmetric:

* A is positive definite iff $\det(A_k) > 0$ for $k = 1, 2, \dots, n$.
* A is negative definite iff $(-1)^k \det(A_k) > 0$ for $k = 1, 2, \dots n$

2015 Final Q6

Let $f(x,y,z) = x^3 + x^2 + y^2 + z^2 - xy + xz$. Find all the critical points of f. Characterize each critical points as a local maximum, a local minimum, or a saddle point.

$f(x,y,z)$ is differentiable for all $(x,y,z) \in \mathbb{R}^3$

① Find the partial derivatives of f and equate them to 0 to find the CPs.

$$\begin{cases} f_x = 3x^2 + 2x - y + z = 0 \longrightarrow 3(2y)^2 + 2(2y) - y - y = 12y^2 + 2y = 0 \\ f_y = 2y - x = 0 \rightarrow x = 2y \\ f_z = 2z + x = 0 \rightarrow x = -2z \end{cases} \quad z = -y$$

$$= y(6y+1) = 0$$

$$\rightarrow y = 0, -1/6$$

$\therefore$ CPs are $(0,0,0)$ and $(-1/3, -1/6, 1/6)$

② Find the Hessian

$$Hf(x,y,z) = \begin{pmatrix} 6x+2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

③ Classify the CPs using the Hessian

$\star$ $Hf(0,0,0) = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\rightarrow A_1 = [2]$, $\det(A_1) = 2 > 0$

$\rightarrow A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $\det(A_2) = 3 > 0$

$\rightarrow A_3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $\det(A_3) = 4 > 0$

$\therefore$ $Hf(0,0,0)$ is positive definite
$\therefore$ $(0,0,0)$ is a local minimum

$\star$ $Hf(-1/3, -1/6, 1/6) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

$\rightarrow A_1 = [0]$, $\det(A_1) = 0$ $= 0$

$\rightarrow A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$, $\det(A_2) = -2 < 0$

$\rightarrow A_3 = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, $\det(A_3) = -4 < 0$

$\therefore$ $Hf(-1/3, -1/6, 1/6)$ is mixed definite
$\therefore$ $(-1/3, -1/6, 1/6)$ is a saddle point

Use Lagrange multipliers to find the constrained critical points of f subject to the given constraints.

① $f(x,y) = x^2+y^2$, $2x+3y=6 \longrightarrow g(x,y) = 2x+3y-6$

$h(x,y,\lambda) = x^2+y^2 - \lambda(2x+3y-6)$

$$\begin{cases} h_x = 2x - 2\lambda = 0 \rightarrow x = \lambda \\ h_y = 2y - 3\lambda = 0 \rightarrow y = \frac{3\lambda}{2} \\ h_\lambda = 6 - 2x - 3y = 0 \longrightarrow 6 - 2\lambda - \frac{9\lambda}{2} = 6 - \frac{13\lambda}{2} = 0 \rightarrow \lambda = \frac{12}{13} \end{cases}$$

$\therefore$ CP is $\left(\frac{12}{13}, \frac{18}{13}\right)$

② $f(x,y)=xy$, $4x^2+9y^2=32 \rightarrow g(x,y)=4x^2+9y^2-32$

$h(x,y,\lambda)=xy-\lambda(4x^2+9y^2-32)$

$$\begin{cases} h_x = y-8x\lambda = 0 \rightarrow \lambda = \dfrac{y}{8x} \\ h_y = x-18y\lambda = 0 \rightarrow \lambda = \dfrac{x}{18y} \\ h_\lambda = 32-4x^2-9y^2 \end{cases}$$

$\dfrac{y}{8x} = \dfrac{x}{18y} \rightarrow 18y^2 = 8x^2 \rightarrow 4x^2 = 9y^2$

or $\lambda = 0$ (contradiction)

$\rightarrow 32-4x^2-4x^2=0$

$8x^2=32$

$x^2=4 \longrightarrow 16=9y^2$

$x=\pm 2$, $y=\pm\frac{4}{3}$

$\therefore$ CPs are $\left(2,\frac{4}{3}\right), \left(2,-\frac{4}{3}\right), \left(-2,\frac{4}{3}\right), \left(-2,-\frac{4}{3}\right)$

Extreme Value Theorem

Let D be a compact set in $\mathbb{R}^n$ and let $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. Then f assumes both a (global) maximum and a (global) minimum on D.

2014 Final Q8

Find the global extrema of $f(x,y,z)=x+y$ on the curve of intersection of the unit sphere, $x^2+y^2+z^2=1$, and the plane, $y+z=1$. Justify your answer including an explanation of why global extrema do exist.

constraints: $g_1(x,y,z)=x^2+y^2+z^2-1$ and $g_2(x,y,z)=y+z-1$

$h(x,y,z,\lambda,\mu) = x+y-\lambda(x^2+y^2+z^2-1)-\mu(y+z-1)$

$$\begin{cases} h_x = 1-2x\lambda = 0 & ① \\ h_y = 1-2y\lambda-\mu = 0 & ② \\ h_z = -2z\lambda-\mu = 0 & ③ \\ h_\lambda = 1-x^2-y^2-z^2 = 0 & ④ \\ h_\mu = 1-y-z = 0 & ⑤ \end{cases}$$

① $\lambda = \dfrac{1}{2x}$

⑤ $y=1-z$

③ $\mu = -2z\lambda$

② $1-2(1-z)\lambda+2z\lambda=0$

$2\lambda(1-2z)=1$

$x=1-2z$

④ $(1-2z)^2+(1-z)^2+z^2=1$

$1-4z+4z^2+1-2z+z^2+z^2=1$

$-6z+6z^2=-1$

$\rightarrow 6z^2-6z+1=0$

$z=\dfrac{6\pm\sqrt{36-24}}{12}=\dfrac{6\pm2\sqrt{3}}{12}=\dfrac{3\pm\sqrt{3}}{6}$

$$x = 1 - 2\left(\frac{3 \pm \sqrt{3}}{6}\right) = \mp\frac{\sqrt{3}}{3}$$

$$y = 1 - \left(\frac{3 \pm \sqrt{3}}{6}\right) = \frac{3 \mp \sqrt{3}}{6}$$

CPs are $\left(-\frac{\sqrt{3}}{3}, \frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right), \left(\frac{\sqrt{3}}{3}, \frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right)$

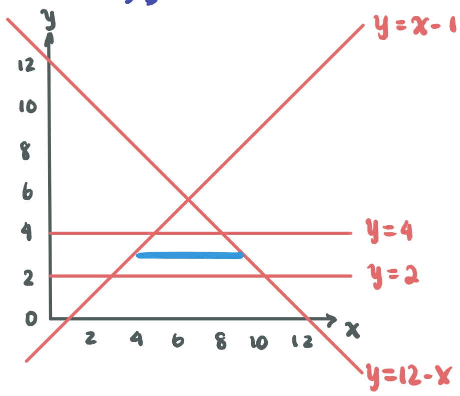
$\rightarrow f\left(-\frac{\sqrt{3}}{3}, \frac{3+\sqrt{3}}{6}, \frac{3-\sqrt{3}}{6}\right) = \frac{1+\sqrt{3}}{2}$ (maximum)

$\rightarrow f\left(\frac{\sqrt{3}}{3}, \frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right) = \frac{1-\sqrt{3}}{2}$ (minimum)

Since f is continuous on $\mathbb{R}^3$ and the intersection is a closed curve which is compact in $\mathbb{R}^3$, the EVT ensures that f will obtain a global maximum and minimum on the intersection.

2015 Final Q10 (a)

Evaluate $\int_D e^{x+y}\,dA$, where D is the region bounded by $y = x-1$ and $y = 12-x$ for $2 \le y \le 4$.



$y - 1 \le x \le 12 - y$

$2 \le y \le 4$

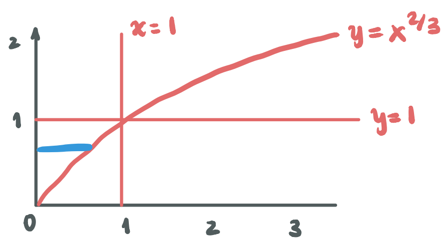
$$\int_D e^{x+y}\,dA = \int_2^4 \int_{y-1}^{12-y} e^{x+y}\,dx\,dy = \int_2^4 e^{x+y}\Big|_{y-1}^{12-y} dy$$

$$= \int_2^4 e^{12} - e^{2y-1}\,dy = \left[e^{12}y - \frac{e^{2y-1}}{2}\right]_2^4$$

$$= 4e^{12} - \frac{e^7}{2} - 2e^{12} + \frac{e^3}{2} = 2e^{12} - \frac{e^7}{2} + \frac{e^3}{2}$$

2015 Final Q10 (b)

Evaluate $\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\,dy\,dx$



$0 \le x \le y^{3/2}$

$0 \le y \le 1$

$$\int_0^1 \int_0^{y^{3/2}} x e^{y^4}\,dx\,dy = \int_0^1 \frac{e^{y^4}}{2}\left[x^2\right]_0^{y^{3/2}} dy$$

Let $u = y^4$

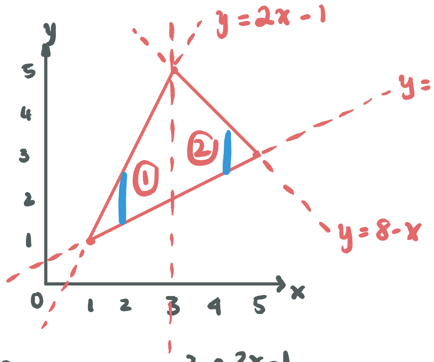
$du = 4y^3\,dy$

$$= \frac{1}{2}\int_0^1 e^{y^4} y^3\,dy = \frac{1}{2} \cdot \frac{1}{4} \int e^u\,du = \frac{1}{8} e^{y^4}\Big|_0^1$$

$$= \frac{1}{8}\left[e - 1\right]$$

2015 Final Q10 (c)

Integrate $f(x,y) = x+1$ over the interior of the triangle with vertices $(1,1),(3,5),(5,3)$



① $\frac{x}{2}+\frac{1}{2} \le y \le 2x-1$

$1 \le x \le 3$

② $\frac{x}{2}+\frac{1}{2} \le y \le 8-x$

$3 \le x \le 5$

$$\int_D x+1\,dA = \int_1^3 \int_{\frac{x}{2}+\frac{1}{2}}^{2x-1} x+1\,dy\,dx + \int_3^5 \int_{\frac{x}{2}+\frac{1}{2}}^{8-x} x+1\,dy\,dx$$

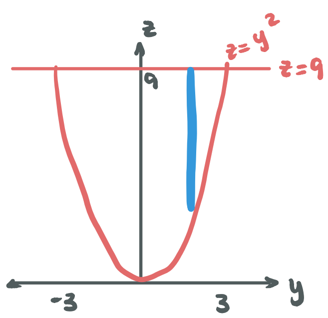
$$= \int_1^3 (x+1)\left(2x-1-\frac{x}{2}-\frac{1}{2}\right)dx + \int_3^5 (x+1)\left(8-x-\frac{x}{2}-\frac{1}{2}\right)dx$$

$$= \int_1^3 (x+1)\left(\frac{3x}{2}-\frac{3}{2}\right)dx + \int_3^5 (x+1)\left(\frac{15}{2}-\frac{3x}{2}\right)dx = \frac{3}{2}\int_1^3 x^2-1\,dx - \frac{3}{2}\int_3^5 x^2-4x-5\,dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3}-x\right]_1^3 - \frac{3}{2}\left[\frac{x^3}{3}-2x^2-5x\right]_3^5 = \frac{3}{2}\left[9-3-\frac{1}{3}+1\right] - \frac{3}{2}\left[\frac{125}{3}-50-25-9+18+15\right]$$

$$= \frac{3}{2}\times\frac{20}{3} - \frac{3}{2}\left[\frac{-28}{3}\right] = 10+14 = 24$$

Find the volume of the region bounded by $x=y^2+z^2$, $z=y^2$, $z=9$ and $x=0$



Fix $0 \le x \le y^2+z^2$

$y^2 \le z \le 9$

$-3 \le y \le 3$

$$V = \int_B 1\,dV = \int_{-3}^3 \int_{y^2}^9 \int_0^{y^2+z^2} dx\,dz\,dy = \int_{-3}^3 \int_{y^2}^9 y^2+z^2\,dz\,dy$$

$$= \int_{-3}^3 \left[y^2 z + \frac{z^3}{3}\right]_{y^2}^9 dy = \int_{-3}^3 9y^2+243-y^4-\frac{y^6}{3}\,dy$$

$$= \left[-\frac{y^7}{21} - \frac{y^5}{5} + 3y^3 + 243y\right]_{-3}^3 = \cdots = \frac{46008}{35}$$