MATB41 Review Seminar

2013 - Q2.(a) i.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

$(x,y) \neq (0,0)$.

$0 \le \left|\frac{y^2}{x^2+y^2}\right| \le 1$, why?? because $x^2 \ge 0$, so $y^2 \le x^2+y^2$..

$$0 \le \left|\frac{xy^2}{x^2+y^2}\right| \le |x|. \to \underbrace{\lim_{(x,y)\to(0,0)} 0}_{=0} \le \lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| \le \underbrace{\lim_{(x,y)\to(0,0)} |x|}_{=0}$$

$\therefore$ By Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| = 0$ //.

2016 - Q2.(a) ii

$$\lim_{(x,y)\to(0,0)} \frac{x^3-x^2y}{\sqrt{x}+\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(x-y)}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x^2\cancel{(\sqrt{x}+\sqrt{y})}(\sqrt{x}-\sqrt{y})}{\cancel{\sqrt{x}+\sqrt{y}}} = \lim_{(x,y)\to(0,0)} x^2(\sqrt{x}-\sqrt{y}) = 0 \,//$$

2012 - Q2 (b)

Define $f: \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \begin{cases} \frac{x\sin(xy)}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$ Is f continuous at $(0,0)$?

For f to be continuous at $(0,0)$ we need $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

$\to$ if $x=0$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{0}{y} = 0$

$$\boxed{\star \lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$\to$ if $x \neq 0$, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2)\left(\frac{\sin(xy)}{xy}\right) = (0)(1) = 0$

$\therefore f$ is cont. at $(0,0)$, because $\lim_{(x,y)\to(0,0)} = f(0,0) = 0$.

2009 - Q3

Characterize and sketch several level curves of the function $f(x,y) = \frac{x^2}{x+y+1}$. Indicate where f is zero, positive, negative and not defined.

Domain is $\{(x,y) \in \mathbb{R}^2 \mid \underbrace{y \neq -x-1}_{\text{undefined line.}}\}$

$C = \frac{x^2}{x+y+1}$,

$Cx + Cy + C = x^2$

$\underline{C=0}$ $\quad x^2 = 0$

$x = 0$.

$\underline{C \neq 0}$ $\quad Cy = x^2 - Cx - C$

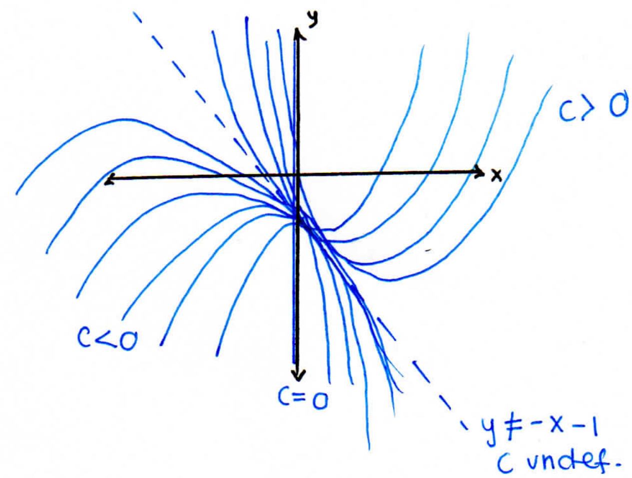
$y = \frac{1}{C}x^2 - x - 1$

$y = \frac{1}{C}\left[x^2 - Cx + \frac{C^2}{4}\right] - 1 - \frac{C}{4}$

$y = \frac{1}{C}\left[x - \frac{C}{2}\right]^2 - \left[\frac{C+4}{4}\right]$

→ $c>0$ parabolas will open upwards.

→ $c<0$ parabolas will open downwards.



2016 - Q5

Let L_1 be the line through $(0,1,1)$ and $(-1,2,1)$; let π be the plane through $(0,1,1)$, $(0,1,0)$ and $(-2,-1,-1)$; and let L_2 be the line orthogonal to π and passing through $(4,0,1)$.

(a) Give both an equation for π and a parametric description for π.

$p=(0,1,0)$. Find two direction vectors $w_1 = (0,1,1)-(0,1,0) = (0,0,1)$

$w_2 = (-2,-1,-1)-(0,1,0) = (-2,-2,-1)$

$\therefore$ Parametric description of π: $(0,1,0)+s(0,0,1)+t(-2,-2,-1)$, $s,t \in \mathbb{R}$.

Find normal vector: $(0,0,1)\times(-2,-2,-1) = \left(\begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ -2 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ -2 & -2 \end{vmatrix} \right)$

$= (2,-2,0)$

The eq. is of the form $2x-2y=d$.

Replacing p. we have $d=-2$.

$\therefore$ Equation of π: $x-y=-1$.

(b) Give parametric descriptions for the lines L_1 and L_2.

$P_1=(0,1,1)$. Direction vector $v_1 = (-1,2,1)-(0,1,1) = (-1,1,0)$.

$\therefore$ Parametric description of L_1: $(0,1,1)+t(-1,1,0)$, $t\in\mathbb{R}$.

$P_2=(4,0,1)$. Direction vector $v_2=(1,-1,0)$ which is the normal vector for π.

$\therefore$ Parametric description of L_2: $(4,0,1)+t(1,-1,0)$, $t\in\mathbb{R}$.

(c) Determine where L_2 meets π.

$(4,0,1)+t(1,-1,0) = (4+t,-t,1)$ satisfies the equation for π when.

$(4+t)-(-t)=-1$

$4+2t=-1$

$2t=-5$

$t=-\frac{5}{2}$

$\therefore$ The point of intersection is $(4-\frac{5}{2}, +\frac{5}{2}, 1)$

$=(\frac{3}{2},\frac{5}{2},1)$ //

(d) Determine if there is a plane containing L_1 and L_2. If there is, find its equation.

L_1 and L_2 are parallel because $v_1=-v_2$, but P_2 does not exist in L_1.

We need 2 direction vectors that can be v_1 and $v_3 = P_2-P_1 = (4,0,1)-(0,1,1) = (4,-1,0)$

The normal of the equation is: $(-1,1,0)\times(4,-1,0) = (0,0,-3)$.

Since P_1 is on the plane, $\therefore$ Equation is $-3z=-3$, $z=1$ //

2013 - Q5

Determine if $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = 3x^2 + 5y^2 + 4xy - 9xz - 8z^2$ is harmonic.

Harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$

★ $f: \mathbb{R}^n \to \mathbb{R}$ is harmonic if $\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$

$\to \frac{\partial f}{\partial x} = 6x + 4y - 9z \to \frac{\partial^2 f}{\partial x^2} = 6$

$\to \frac{\partial f}{\partial y} = 10y + 4x \to \frac{\partial^2 f}{\partial y^2} = 10$

$\to \frac{\partial f}{\partial z} = -9x - 16z \to \frac{\partial^2 f}{\partial z^2} = -16$

$6 + 10 - 16 = 0$

$\therefore$ f is harmonic.

2012 - Q5

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = 2x^2 + 2xz + y^2 + 4y + yz$.

(a) What is the rate of change in f if you move from (1,0,1) towards (1,2,3)?

$V = (1,2,3) - (1,0,1) = (0,2,2)$

$D_{(0,2,2)} f(1,0,1) = \nabla f(1,0,1) \cdot \frac{(0,2,2)}{\|(0,2,2)\|} = \frac{(6,5,2)\cdot(0,2,2)}{\sqrt{4+4}}$

$\nabla f = (4x + 2z, 2y + 4 + z, 2x + y)$

$\nabla f(1,0,1) = (6,5,2)$

$= \frac{14}{\sqrt{8}} = \frac{7}{\sqrt{2}}$ //.

★ The r.o.c. in f from P_1 to P_2 is the directional derivative $D_v f(P_1)$ where $V = P_2 - P_1$.

(b) What is the direction of the maximum rate of increase in f at (1,0,1)? What is the magnitude of the maximum increase?

$\therefore$ Direction of the max rate of inc. in f at (1,0,1) is $\nabla f(1,0,1) = (6,5,2)$ //

$\therefore$ Maximum increase is $\|\nabla f(1,0,1)\|$
$= \|(6,5,2)\|$
$= \sqrt{65}$ //

★ The direction of the maximum rate of increase is the direction of the gradient of f at $P \to \nabla f(P)$

★ The magnitude of the maximum inc. $\|\nabla f(P)\|$

(c) Find the critical points of f.

f is a polynomial $\to$ differentiable $\forall (x,y,z) \in \mathbb{R}^3$.

$\nabla f(x,y,z) = (4x + 2z, 2y + z + 4, 2x + y) = 0.$

$$\begin{cases} 4x + 2z = 0 \to z = -2x \\ 2y + z = -4 \\ 2x + y = 0 \to y = -2x \end{cases}$$

$-4x - 2x = -4$
$6x = 4$
$x = \frac{2}{3}$

$y = -\frac{4}{3}$
$z = -\frac{4}{3}$

$\therefore$ The only critical point is $\left(\frac{2}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$.

2016 - Q6 (a)

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \frac{x+y}{x^2}$. Find an equation for the tangent plane to the graph of $z = f(x,y)$ at the point $(2,3,f(2,3))$.

☆ When $p = (a,b,f(a,b))$ the eq. of the tangent plane is $z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

$f_x = \frac{x^2 - 2x(x+y)}{x^4} = \frac{-(x+2y)}{x^3}$

$\to \frac{\partial f}{\partial x}(2,3) = -1$

$f_y = \frac{x^2}{x^4} = \frac{1}{x^2}$

$\to \frac{\partial f}{\partial y}(2,3) = \frac{1}{4}$

$f(2,3) = \frac{5}{4}$

$\therefore z = \frac{5}{4} - (x-2) + \frac{1}{4}(y-3)$

$4z = 5 - 4x + 8 + y - 3$

$10 = 4x - y + 4z$ //

2016 - Q7 (a)

Compute an equation for the tangent plane of the surface $x^3 + xy^2 + x^2 + y^2 + 3z^2 = 3$ at the point $(-1,2,1)$.

$g(x,y,z) = x^3 + xy^2 + x^2 + y^2 + 3z^2 - 3$.

Normal to the surface is $\nabla g(x,y,z) = (3x^2 + y^2 + 2x,\ 2xy + 2y,\ 6z)$

$\nabla g(-1,2,1) = (3+4-2,\ -4+4,\ 6)$

$= (5,0,6) \to$ Tangent plane normal.

Equation will be: $5x + 6z = k$

$(-1,2,1)$ is a point on the plane $\to -5 + 6 = 1 = k$.

$\therefore 5x + 6z = 1$ //

2012 - Q7 (b)

Let $f:\mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x,y,z) = (xy^2, yz^2, x^2z)$ and $g:\mathbb{R}^3 \to \mathbb{R}^4$ be given by $g(x,y,z) = (xz, xyz, x+z, y^2)$. Use chain rule to compute $D(g\circ f)(x,y,z)$.

☆ $D(g\circ f)(x,y,z) = [Dg(f(x,y,z))][Df(x,y,z)]$

$$Df = \begin{pmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \end{pmatrix},\quad Dg = \begin{pmatrix} z & 0 & x \\ yz & xz & xy \\ 1 & 0 & 1 \\ 0 & 2y & 0 \end{pmatrix}$$

$$Dg(f(x,y,z)) = Dg(xy^2, yz^2, x^2z) = \begin{pmatrix} x^2z & 0 & xy^2 \\ x^2yz^3 & x^3y^2z & xy^3z^2 \\ 1 & 0 & 1 \\ 0 & 2yz^2 & 0 \end{pmatrix}$$

$$\therefore D(g\circ f)(x,y,z) = \begin{pmatrix} x^2z & 0 & xy^2 \\ x^2yz^3 & x^3y^2z & xy^3z^2 \\ 1 & 0 & 1 \\ 0 & 2yz^2 & 0 \end{pmatrix} \begin{pmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \end{pmatrix}$$

$$= \begin{pmatrix} 3x^2y^2z & 2x^3yz & x^3y^2 \\ 3x^2y^3z^3 & 3x^3y^2z^3 & 3x^3y^3z^2 \\ y^2 + 2xz & 2xy & x^2 \\ 0 & 2yz^4 & 4y^2z^3 \end{pmatrix}$$

2009- Q9.

Let $z = f(x,y)$ be of class C^2. Putting $x = 2u - 3v$ and $y = 4u + 5v$ makes z into a function of u and v. Compute a furmula for $\frac{\partial^2 z}{\partial v \partial u}$ in terms of the partial derivatives of z with respect to x and y.

$\rightarrow \frac{\partial x}{\partial u} = 2$ $\rightarrow \frac{\partial x}{\partial v} = -3$

$\rightarrow \frac{\partial y}{\partial u} = 4$ $\rightarrow \frac{\partial y}{\partial v} = 5$

$$\frac{\partial^2 f}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right]$$

$$= \frac{\partial}{\partial v}\left[2\frac{\partial f}{\partial x} + 4\frac{\partial f}{\partial x}\right] = 2\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) + 4\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{2\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) + 4\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right) = \frac{2\partial}{\partial x}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right] + \frac{4\partial}{\partial y}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\right.$$

$$= \frac{2\partial}{\partial x}\left[-3\frac{\partial f}{\partial x} + \frac{5\partial f}{\partial y}\right] + \frac{4\partial}{\partial y}\left[-\frac{3\partial f}{\partial x} + \frac{5\partial f}{\partial y}\right]$$

$$= -\frac{6\partial^2 f}{\partial x^2} + \frac{10\partial^2 f}{\partial x \partial y} - \frac{12\partial^2 f}{\partial y \partial x} + \frac{20\partial^2 f}{\partial y^2}$$

$$= -\frac{6\partial^2 f}{\partial x^2} - \frac{2\partial^2 f}{\partial x \partial y} + \frac{20\partial^2 f}{\partial y^2} \quad //$$

2012 - Q9.

Give the 6^{th} degree Taylor polynomial about the origin of $f(x,y) = \cos(xy)\ln(1-x^2)$

- $\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}, \quad |t| < \infty$.

$$\rightarrow \cos(xy) = \sum_{k=0}^{\infty} (-1)^k \frac{(xy)^{2k}}{(2k)!} = 1 - \frac{x^2y^2}{2!} + \frac{x^4y^4}{4!} - \dots, \quad |xy| < \infty$$

- $\ln(1+t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}, \quad |t| < 1$.

$$\rightarrow \ln(1-x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(-x^2)^{k+1}}{k+1} = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \dots, \quad |x| < 1.$$

$$T = \left(1 - \frac{x^2y^2}{2!} + \frac{x^4y^4}{4!} - \dots\right)\left(-x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \dots\right)$$

$$= -x^2 - \frac{x^4}{2} - \frac{x^6}{3} + \frac{x^4y^2}{2!} + \frac{x^6y^2}{2\cdot 2!} + \frac{x^8y^2}{3!} - \frac{x^6y^4}{4!} - \frac{x^8y^4}{2\cdot 4!} - \frac{x^{10}y^4}{3\cdot 4!} + \dots$$

$$T_6 = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} + \frac{x^4y^2}{2}. \quad //$$