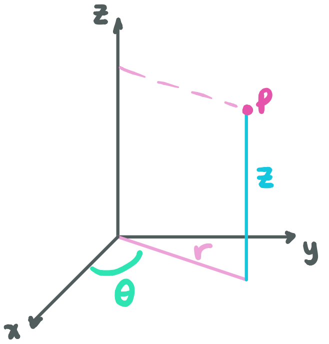
MATB41 - Midterm Review Seminar

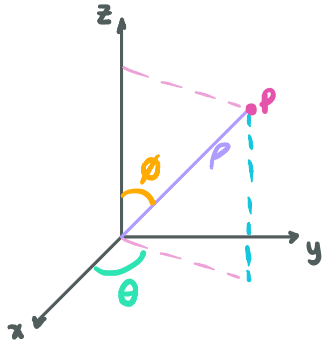
Cylindrical Coordinates:



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$0 \le r \le \infty$
$0 \le \theta \le 2\pi$

Spherical Coordinates:



$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$0 \le \rho \le \infty$
$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

(Q3 c) - 2018) Find an equation in spherical coordinates for the cylinder $x^2+y^2=4$ in $\mathbb{R}^3$.

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = 4$
$\rightarrow \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) = 4$
$\rightarrow \rho^2\sin^2\phi = 4$
$\rightarrow \rho = \dfrac{2}{\sin\phi}$

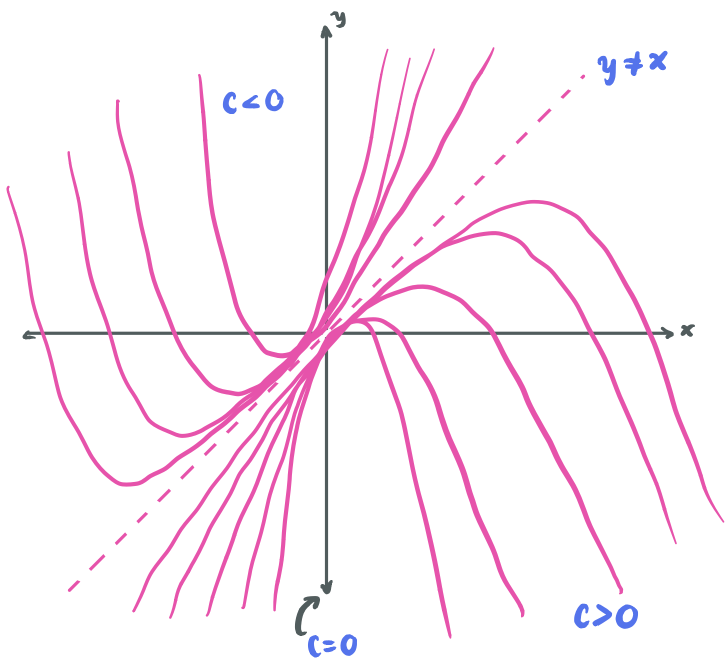
(Q4 - 2015) Characterize and sketch several level curves of the function $f(x,y) = \dfrac{x^2}{x-y}$. Indicate where f is zero, positive, negative and not defined

Domain is $\{(x,y) \in \mathbb{R}^2 \mid y \neq x\}$

Let $\dfrac{x^2}{x-y} = c$
$\rightarrow x^2 = c(x-y)$

When $c=0$
$x=0$, so y-axis $(y\neq 0)$

When $c \neq 0$
$\rightarrow x^2 = cx - cy$
$\rightarrow cy = cx - x^2$
$\rightarrow y = x - \dfrac{x^2}{c} \rightarrow y = x\left(1-\dfrac{x}{c}\right)$ roots $x=0, c$
$\rightarrow y = -\dfrac{1}{c}(x^2 - cx)$
$\rightarrow y = -\dfrac{1}{c}\left(x^2 - cx + \dfrac{c^2}{4} - \dfrac{c^2}{4}\right)$
$\rightarrow y = -\dfrac{1}{c}\left(x-\dfrac{c}{2}\right)^2 + \dfrac{c}{4}$



$c<0$ upwards
$c>0$ downwards

(Q6 - 2015)

(a) Determine the rate of change in $f(x,y,z) = y - x^2 + z^2$ as you move from $(-1,0,2)$ towards $(2,4,2)$

Directional derivative:

$$D_v f(p) = \nabla f(p) \cdot \frac{v}{\|v\|}$$

$v = (2,4,2) - (-1,0,2) = (3,4,0)$

$\nabla f(x,y,z) = (-2x, 1, 2z)$
$\nabla f(-1,0,2) = (2,1,4)$
$\|(3,4,0)\| = \sqrt{9+16} = 5$
$(2,1,4)\cdot(3,4,0) = 10$

$\therefore D_{(3,4,0)} f(-1,0,2) = \frac{10}{5} = 2$

(b) Compute the directional derivative of $f(x,y,z) = x^2y^3z^2$ at the point $(2,1,-1)$ in the direction of the upward normal for the plane $2x + y - 2z = -7$

$v = k(2,1,-2)$, $k \in \mathbb{R}$. z has to be positive, so let $k = -1$
$v = (-2,-1,2)$

$\nabla f(x,y,z) = (2xy^3z^2, 3x^2y^2z^2, 2x^2y^3z)$
$\nabla f(2,1,-1) = (4,12,-8)$
$\|(-2,-1,2)\| = \sqrt{4+1+4} = 3$
$(4,12,-8)\cdot(-2,-1,2) = -8-12-16 = -36$

$\therefore D_{(-2,-1,2)} f(2,1,-1) = -\frac{36}{3} = -12$

(Q7 - 2015) Let π be the plane in $\mathbb{R}^3$ passing through the points $(-1,0,2)$, $(1,3,1)$ and $(2,1,-1)$

(a) Give an equation for π

Let $p = (-1,0,2)$. We have two vectors: $v_1 = (1,3,1) - (-1,0,2) = (2,3,-1)$
$v_2 = (2,1,-1) - (-1,0,2) = (3,1,-3)$

To find the normal of π we need to $v_1 \times v_2 = (-9+1, 6-3, 2-9) = (-8, 3, -7)$

$\Rightarrow$ The equation of the plane is $-8x + 3y - 7z = d$.
If we input p, we get $8 - 14 = -6 = d$

$\therefore$ The equation of π is $8x - 3y + 7z = 6$

(b) Give a parametric description of the line ℓ through $(0,0,1)$ and orthogonal to π. Where does ℓ meet π?

$\therefore$ The parametric description is $(0,0,1) + t(8,-3,7)$, $t \in \mathbb{R}$

The line and plane will intersect when $(8t, -3t, 1+7t)$ satisfies π:

$8(8t) - 3(-3t) + 7(1+7t) = 6$

$\rightarrow 64t + 9t + 7 + 49t = 6$

$\rightarrow \quad 122t = -1$

$\rightarrow \quad t = \frac{-1}{122}$

$\therefore$ l meets π at $\left(\frac{-8}{122}, \frac{3}{122}, \frac{115}{122}\right) = \left(\frac{-4}{61}, \frac{3}{122}, \frac{115}{122}\right)$

(Q8-2015) Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $f(x,y,z) = (e^{yz}, xy, x^2z)$ and let $g: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by $g(x,y,z) = (y^2+z, xz^2)$. Compute $D(g\circ f)(x,y,z)$ using chain rule.

$D(g\circ f)(x,y,z) = Dg(f(x,y,z))\, Df(x,y,z)$

$\Rightarrow Dg(x,y,z) = \begin{pmatrix} 0 & 2y & 1 \\ z^2 & 0 & 2xz \end{pmatrix}$

$\Rightarrow Dg(f(x,y,z)) = Dg(e^{yz}, xy, x^2z) = \begin{pmatrix} 0 & 2xy & 1 \\ x^4z^2 & 0 & 2x^2ze^{yz} \end{pmatrix}$

$\Rightarrow Df(x,y,z) = \begin{pmatrix} 0 & ze^{yz} & ye^{yz} \\ y & x & 0 \\ 2xz & 0 & x^2 \end{pmatrix}$

$\therefore D(g\circ f)(x,y,z) = Dg(f(x,y,z))\, Df(x,y,z)$

$= \begin{pmatrix} 0 & 2xy & 1 \\ x^4z^2 & 0 & 2x^2ze^{yz} \end{pmatrix} \begin{pmatrix} 0 & ze^{yz} & ye^{yz} \\ y & x & 0 \\ 2xz & 0 & x^2 \end{pmatrix}$

$= \begin{pmatrix} 2xy^2 + 2xz & 2x^2y & x^2 \\ 4x^3z^2e^{yz} & x^4z^3e^{yz} & x^4yz^2e^{yz} + 2x^4ze^{yz} \end{pmatrix}$

Taylor Series

(x,y) around $(0,0)$:

$$F(x,y) = F(0,0)$$
$$+ x\frac{\partial F}{\partial x}(0,0) + y\frac{\partial F}{\partial y}(0,0)$$
$$+ \frac{1}{2!}\left[x^2 \frac{\partial^2 F}{\partial x^2}(0,0) + 2xy\frac{\partial^2 F}{\partial x \partial y}(0,0) + y^2\frac{\partial^2 F}{\partial y^2}(0,0)\right]$$
$$+ \cdots$$

else:

$$F(x,y) = F(a,b) + \frac{\partial F}{\partial x}(x-a) + \frac{\partial F}{\partial y}(y-b) + \frac{1}{2!}\left[\frac{\partial^2 F}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 F}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 F}{\partial y^2}(y-b)^2\right] + \dots$$

Common series around $a=0$ are:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad |x| < \infty \qquad \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \quad |x| < 1$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad |x| < \infty \qquad \ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}, \quad |x| < 1$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad |x| < \infty \qquad \arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}, \quad |x| < 1$$

Find the 3^{rd} degree Taylor polynomial about the origin of $f(x,y) = (\sin x)\ln(1+y)$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots$$

$$T = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \dots\right)$$

$$\therefore T_3 = xy - \frac{1}{2}xy^2.$$

Directly compute the second degree Taylor polynomial about the point $(1,0)$ for $f(x,y) = e^{(x-1)^2}\cos y$

$\rightarrow f(1,0) = e^0\cos(0) = 1$

$\rightarrow \frac{\partial f}{\partial x}(x,y) = 2(x-1)e^{(x-1)^2}\cos y \qquad \frac{\partial f}{\partial x}(1,0) = 2(0)e^0\cos(0) = 0$

$\frac{\partial f}{\partial y}(x,y) = -e^{(x-1)^2}\sin y \qquad \frac{\partial f}{\partial y}(1,0) = e^0\sin(0) = 0$

$\rightarrow \dfrac{\partial^2 f}{\partial x^2}(x,y) = \left(2e^{(x-1)^2} + 4(x-1)^2 e^{(x-1)^2}\right)\cos y$ $\quad$ $\dfrac{\partial^2 f}{\partial x}(1,0) = (2e^0 + 4(0)e^0)\cos(0) = 2$

$\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = -2(x-1)e^{(x-1)^2}\sin y$ $\quad$ $\dfrac{\partial^2 f}{\partial x \partial y}(1,0) = -2(0)e^0 \sin(0) = 0$

$\dfrac{\partial^2 f}{\partial y^2}(x,y) = -e^{(x-1)^2}\cos y$ $\quad$ $\dfrac{\partial^2 f}{\partial y^2}(1,0) = -e^0\cos(0) = -1$

$$\therefore T_2 = f(1,0) + \left[\frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)\,y\right] + \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(1,0)(x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,0)(x-1)y + \frac{\partial^2 f}{\partial y^2}y^2\right]$$

$$= 1 + \frac{1}{2!}\left[2(x-1)^2 - y^2\right]$$

$$= 1 + (x-1)^2 - \frac{y^2}{2}$$