MATB44 - Tutorial 6 (IC 320 Fridays 10-11am)

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Week 1 - Sept. 6th

No tutorials

Week 2 - Sept. 13th

ODE: $F(t, x, x^{(1)}, \ldots, x^{(k)}) = 0$

t - independent variable (time)
x - dependent variable (space)
x is a function of t i.e. $x(t)$

The order is the highest derivative that appears in F.

The system is linear if we can rewrite it to separate out the $x_\ell^{(j)}$'s as follows:

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=1}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

Ex: $\underbrace{(4+t^2)x' + 2tx}= 4t$

By product rule: $\frac{d}{dt}\left[(4+t^2)x\right] = 2tx + (4+t^2)x'$

$\Rightarrow \frac{d}{dt}\left[(4+t^2)x\right] = 4t$

$\Rightarrow (4+t^2)x = 2t^2 + C$

$\Rightarrow x(t) = \frac{2t^2 + C}{4+t^2}$

The system is homogeneous if $g_i(t) = 0$

If there is no direct dependance on t, the system is autonomous. Form $x' = f(x)$

Ex: Classify the following equations:

* $x' + \frac{1}{2}x = \frac{1}{2}e^{t/3}$ 1st order, linear

* $3 + xx' = t - x$ 1st order, non linear

* $x''+5x'+6x=0$ 2nd order, linear, homogeneous, autonomous
* $x'=t\sin x$ 1st order, non linear
* $x''-x=0$ 2nd order, linear, homogeneous, autonomous
* $tx'+2x=4t^2$ 1st order, linear
* $x'-2x+t=4$ 1st order, linear.

Ex: Solve $x'=x^2$ with $x_0=x(0)>0$

① $\frac{dx}{dt}=x^2 \Rightarrow \frac{dx}{x^2}=dt$

$$\Rightarrow \int \frac{dx}{x^2}=\int dt$$

$$\Rightarrow -\frac{1}{x}=t+C_1$$

$$\Rightarrow x=\frac{1}{-C_1-t}$$

When $x(0)=-\frac{1}{C_1}=x_0$ so $x=\frac{1}{\frac{1}{x_0}-t}$

② $f(x)=x^2$, a maximal nonzero interval is $(x_1,x_2)=(0,\infty)$

$$F(x)=\int_{x_0}^{x}\frac{dy}{y^2}=-\frac{1}{y}\Big|_{y=x_0}^{y=x}=-\frac{1}{x}+\frac{1}{x_0}$$

$\phi(t)=F^{-1}(t)$

$T_+=\lim_{x\to x_2}F(x)=\lim_{x\to\infty}-\frac{1}{x}+\frac{1}{x_0}=\frac{1}{x_0}$, so ϕ is defined for all $t>0$

$$t=F(F^{-1}(t))=\frac{-1}{F^{-1}(t)}+\frac{1}{x_0}$$

$$\Rightarrow \frac{1}{F^{-1}(t)}=\frac{1}{x_0}-t$$

$$\Rightarrow F^{-1}(t)=\frac{1}{\frac{1}{x_0}-t}$$

$$\Rightarrow \phi(t)=\frac{1}{\frac{1}{x_0}-t}$$

■

Week 3 - Sept. 20th

Office Hours: Wed. 10-11 am IC404

A particular solution is a solution that has no arbitrary constant. A general solution is a k-parameter family of solutions that contain every particular solution

A separable first-order ODE is an ODE that can be rewritten as:

$$\boxed{\dot{x}(t,x) = g(t)f(x)} \quad (\text{or } g(t)\,dt + f(x)\,dx = 0)$$

$$\Rightarrow \frac{dx}{dt} = g(t)f(x)$$

$$\Rightarrow \frac{dx}{f(x)} = g(t)\,dt$$

$$\Rightarrow \boxed{\int \frac{dx}{f(x)} = \int g(t)\,dt}, \quad f(x) \neq 0$$

Ex: Show that the equation $\dot{x}(t,x) = \frac{t^2}{1-x^2}$ is separable and solve.

Let $f(x) = \frac{1}{1-x^2}$ and $g(t) = t^2$

$$\Rightarrow \int 1-x^2\,dx = \int t^2\,dt$$

$$x - \frac{x^3}{3} = \frac{t^3}{3} + C_1$$

$$3x - x^3 - t^3 = C \quad , C \in \mathbb{R}$$

Ex: Solve the equation $\frac{dx}{dt} = \frac{4t - t^3}{4 + x^3}$ and find the solution passing through the point $(0,1)$

$$\Rightarrow (4+x^3)\,dx = (4t - t^3)\,dt$$

$$\int (4+x^3)\,dx = \int (4t - t^3)\,dt$$

$$4x + \frac{x^4}{4} = 2t^2 - \frac{t^4}{4} + C_1$$

$$\Rightarrow 16x + x^4 - 8t^2 + t^4 = C \quad , C \in \mathbb{R},$$

When $(0,1) \rightarrow 16 + 1 = 17$

$$\therefore 16x + x^4 - 8t^2 + t^4 = 17$$

Let $z=f(x,y)$ be a function of x and y. $f(x,y)$ is homogeneous of order n if it can be written as $\boxed{f(x,y)=x^n g(u)}$, where $u=\frac{y}{x}$ or $f(x,y)=y^n g(u)$, where $u=\frac{x}{y}$.

$\boxed{P(x,y)dx+Q(x,y)dy=0}$ where $P(x,y)$ and $Q(x,y)$ are the homogeneous coefficients and can be solved by substituting $y=ux$, $dy=udx+xdu$

Ex: Find the general solution of $txx'=t^2+2x^2$

$$\Rightarrow \frac{1}{t^2}\left[txx'=t^2+2x^2\right]$$

$$\frac{x}{t}x'=1+2\frac{x^2}{t^2}$$

Let $u=\frac{x}{t}$ so $x=ut$, $x'=u+tu'$

$$\Rightarrow u(u+tu')=1+2u^2$$
$$u^2+tu'u=1+2u^2$$
$$tu'u=1+u^2$$
$$\frac{u'u}{1+u^2}=\frac{1}{t}$$

$\rightarrow$ OMG! This is separable!

$$\int\frac{uu'}{1+u^2}=\int\frac{1}{t}dt$$

Let $v=1+u^2$, $dv=2udu$

$$\frac{1}{2}\int\frac{dv}{v}=\ln|t|+C_1$$
$$\frac{1}{2}\ln|1+u^2|=\ln|t|+C$$
$$\therefore \frac{1}{2}\ln\left|1+\frac{x^2}{t^2}\right|-\ln|t|=C$$

A differential expression $\boxed{P(x,y)dx+Q(x,y)dy}$ is called an exact differential if it is the total differential of some function $f(x,y)$.

i.e. if $\boxed{P(x,y)=\frac{\partial}{\partial x}f(x,y)}$ and $\boxed{Q(x,y)=\frac{\partial}{\partial y}f(x,y)}$.

If we can find $f(x,y)$, $f(x,y)=C$ is the 1-parameter family of solutions.

$P(x,y)dx+Q(x,y)dy=0$ is exact iff $\frac{\partial P(x,y)}{\partial y}=\frac{\partial Q(x,y)}{\partial x}$.

Ex: Find the general solution of $y'=-\frac{1+2xy^2}{1+2x^2y}$

$$\Rightarrow (1+2x^2y)y'=-(1+2xy^2)$$
$$(1+2xy^2)+(1+2x^2y)y'=0$$

Is it exact? $P(x,y)=1+2xy^2$ $\quad Q(x,y)=1+2x^2y$

$$\frac{\partial P(x,y)}{\partial y}=4xy \iff \frac{\partial Q(x,y)}{\partial x}=4xy \quad ✓$$

let's take $P(x,y)$, we know $P(x,y) = \frac{\partial}{\partial x} f(x,y)$

$$\Rightarrow \int P(x,y)\,dx = f(x,y)$$

$$\int 1+2xy^2\,dx = x + x^2y^2 + g(y) = f(x,y)$$

$$\Rightarrow \frac{\partial}{\partial y} f(x,y) = 2x^2y + g'(y) = Q(x,y) = 1 + 2x^2y$$

So $g'(y) = 1 \rightarrow g(y) = y$

$$\therefore x + x^2y^2 + y = C$$

Week 4 - Sept. 27th

An integrating factor (IF) will convert an inexact ODE $P(x,y)dx + Q(x,y)dy = 0$ into an exact ODE $IF\, P(x,y)dx + IF\, Q(x,y)dy = 0$

Given $\frac{dy}{dx} + P(x)y = Q(x)$ a known IF is $e^{\int P(x)dx}$

Ex: Find the general solution of $\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$

$P(x) = \frac{3}{x}$, $IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$$\Rightarrow x^3\frac{dy}{dx} + x^3 \cdot \frac{3y}{x} = x^3\frac{e^x}{x^3}$$

$$\Rightarrow x^3y' + 3x^2y = e^x$$

$$\Rightarrow \int (x^3y' + 3x^2y)\,dx = \int e^x dx$$

$$\Rightarrow x^3y = e^x + C$$

$$\Rightarrow y = \frac{e^x + C}{x^3}$$

Ex: Solve $6y' - 2y = xy^4$, $y(0) = -2$.

$\Rightarrow 6y^{-4}y' - 2y^{-3} = x$ let $u = y^{-3}$

$du = -3y^{-4}y'$

$\Rightarrow -2u' - 2u = x$ or $u' + u = -\frac{x}{2}$

$P(x) = 1$, $IF = e^{\int P dx} = e^x$

$\Rightarrow u'e^x + ue^x = -\frac{e^x x}{2}$

$\Rightarrow \int (u'e^x + ue^x)dx = -\frac{1}{2}\int xe^x dx$

$\Rightarrow ue^x = -\frac{1}{2}(xe^x - e^x) + c$

$\Rightarrow u = -\frac{1}{2}(x-1) + ce^{-x}$

$\Rightarrow y^{-3} = -\frac{1}{2}(x-1) + ce^{-x}$

$\Rightarrow 2y^{-3} = (1-x) + 2ce^{-x}$

$\Rightarrow y^3 = \frac{2}{(1-x)+2ce^{-x}}$

$\Rightarrow y = \left[\frac{2}{\left[(1-x) - \frac{5}{4}e^{-x}\right]}\right]^{1/3}$

Side note:

$\int xe^x dx$ $\quad u = x \quad dv = e^x dx$; $du = dx \quad v = e^x$

$= xe^x - \int e^x dx = xe^x - e^x + C$.

IVP

$\Rightarrow (-2)^3 = \frac{2}{1+2c}$

$\Rightarrow -8 - 16c = 2$

$\Rightarrow c = -\frac{5}{8}$

Bernoulli Equation: given $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

multiply by $(1-n)y^{-n} \Rightarrow (1-n)y^{-1}\frac{dy}{dx} + (1-n)y^{1-n}P(x) = (1-n)Q(x)$

Substitute $u = y^{1-n}$

$du = (1-n)y^{-n}dy \quad \Rightarrow \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$

$IF\ e^{\int (1-n)P(x)dx}$

Let X be a real **vector space**. A **norm** on X is a map $\|\cdot\|: X \to [0,\infty)$ s.t.

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \neq 0$

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$

Together $(X, \|\cdot\|)$ is a **normed vector space**. A sequence of vectors x_n **converges** to x if $\lim_{n\to\infty} \|x_n - x\| = 0$.

A contraction is a mapping $K: C \subseteq X \to C$ where there exists a contraction constant $\theta \in [0,1)$ s.t. $\|K(x) - K(y)\| \leq \theta \|x-y\|$, $x, y \in C$.

Note: $K^n(x) = K(K^{n-1}(x))$ $\quad K^0(x) = x$

Ex: Let $K(x) = 40 + \frac{x}{3}$. What is the contraction constant? Is $K(x)$ a contraction on $C=[0,90]$? on $C=[0,30]$?

$$\|K(x) - K(y)\| = \|40 + \frac{x}{3} - 40 - \frac{y}{3}\| = \|\frac{x}{3} - \frac{y}{3}\| = \frac{1}{3}\|x-y\|$$

$\therefore \theta = \frac{1}{3}$

$C[0,90]$ ✓ $\quad K(0) = 40 \in [0,90]$

$C[0,30]$ X $\quad K(0) = 40 \notin [0,30]$

Consider an initial value problem (IVP) $\dot{x} = f(t,x)$, $x(t_0) = x_0$, where $x, t \in \mathbb{R}$ and $f \in C(U, \mathbb{R})$ where $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Let's define Picard Iteration by a map $K: C(U,\mathbb{R}) \to C(U,\mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

and the Picard iterates.

$x_0(t) = x_0$ (the constant function through the scalar x_0)

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds.$$

$\vdots$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\,ds$$

Ex: Calculate the Picard Iterates x_0, x_1, x_2. for $f(t,x) = 3-2x$, $x(t_0) = x_0$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

$$K^{m+1}(x)(t) = x_0 + \int_{t_0}^{t} f(s, x_m(s))\,ds = x_0 + \int_{t_0}^{t} (3 - 2x_m(s))\,ds.$$

$$\Rightarrow x_0(t) = x_0$$

$$\Rightarrow x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} 3 - 2x_0(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} 3 - 2x_0\,ds$$

$$= x_0 + (3-2x_0)(t-t_0)$$

$$= x_0 + t(3-2x_0) - t_0(3-2x_0)$$

$$\Rightarrow x_2(t) = K^2(x_0(t)) = x_0 + \int_{t_0}^{t} 3 - 2x_1(s)\,ds$$

$$= x_0 + \int_{t_0}^{t} 3 - 2\big(x_0 + s(3-2x_0) - t_0(3-2x_0)\big)\,ds$$

$$= x_0 + \left[3 - 2x_0 + 2t_0(3-2x_0)\right](t-t_0) - 2(3-2x_0)\int_{t_0}^{t} s\,ds$$

$$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)\,s^2\Big|_{t_0}^{t}$$

$$= x_0 + (3-2x_0)(1+2t_0)(t-t_0) - (3-2x_0)(t^2 - t_0)$$

Week 5 - Oct. 4th

$\dot{x} = f(t,x)$, $x(t_0) = x_0$, $x: \mathbb{R} \to \mathbb{R}^n$

Let $K: C(U,\mathbb{R}^n) \to C(U,\mathbb{R}^n)$, $U \subseteq \mathbb{R}^{n+1}$ an open set

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

Then the sequence $x_0(t) = x_0$, $x_m = K(x_{m-1})$ converges to the solution x.

Ex: $\dot{x}(t) = \begin{pmatrix} t+y \\ y^2+z^2 \end{pmatrix}$, $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{matrix} y(0)=1 \\ z(0)=0 \end{matrix}$

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s + (y(0))\,ds \\ \int_0^t (y(0))^2 + (z(0))^2\,ds \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s+1\,ds \\ \int_0^t ds \end{pmatrix} = \begin{pmatrix} \frac{t^2}{2} + t + 1 \\ t \end{pmatrix} \begin{matrix} \to y_1 \\ \to z_1 \end{matrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \int_0^t s + \left(\frac{s^2}{2} + s + 1\right) ds^{(1)} \\ \int_0^t \left(\frac{s^2}{2} + s + 1\right)^2 + s^2\,ds^{(2)} \end{pmatrix} = \begin{pmatrix} 1 + \frac{t^3}{6} + t^2 + t \\ \frac{t^5}{20} + \frac{t^4}{4} + t^3 + t^2 + t \end{pmatrix}$$

(1) $\int_0^t \frac{s^2}{2} + 2s + 1\,ds = \left[\frac{s^3}{6} + s^2 + s\right]_0^t$

(2) $\int_0^t \left(\frac{s^4}{4} + s^3 + 2s^2 + 2s + 1\right) + s^2\,ds = \left[\frac{s^5}{20} + \frac{s^4}{4} + s^3 + s^2 + s\right]_0^t$

Ex: $\dot{x}(t) = \begin{pmatrix} 2z+t^2 \\ t+y \end{pmatrix}$, $x(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{matrix} y(1)=1 \\ z(1)=1 \end{matrix}$

$$x_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t 2+s^2 ds \\ \int_1^t s+1 ds \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \left[2s+\frac{s^3}{3}\right]_1^t \\ \left[\frac{s^2}{2}+s\right]_1^t \end{pmatrix} = \begin{pmatrix} 1+2t+\frac{t^3}{3}-2-\frac{1}{3} \\ 1+\frac{t^2}{2}+t-\frac{1}{2}-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2t+\frac{t^3}{3}-\frac{4}{3} \\ \frac{t^2}{2}+t-\frac{1}{2} \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t s^2+2s-1+s^2 ds \\ \int_1^t s+2s+\frac{s^3}{3}-\frac{4}{3} ds \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_1^t 2s^2+2s-1 ds \\ \int_1^t 3s+\frac{s^3}{3}-\frac{4}{3} ds \end{pmatrix}$$

$$= \begin{pmatrix} 1+\left[\frac{2s^3}{3}+s^2-s\right]_1^t \\ 1+\left[\frac{3s^2}{2}+\frac{s^4}{12}-\frac{4s}{3}\right]_1^t \end{pmatrix} = \begin{pmatrix} 1+\frac{2t^3}{3}+t^2-t-\frac{2}{3}-1+1 \\ 1+\frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}-\frac{3}{2}-\frac{1}{12}+\frac{4}{3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2t^3}{3}+t^2-t+\frac{1}{3} \\ \frac{3t^2}{2}+\frac{t^4}{12}-\frac{4t}{3}+\frac{3}{4} \end{pmatrix}$$

Picard-Lindelöf

Ex: $\dot{x} = x^{1/3}+t$, $x(1)=0$

$V_0 = [0,2] \times [-1,1]$

$$\frac{|f(t,x)-f(t,y)|}{|x-y|} = \frac{|x^{1/3}+t-y^{1/3}-t|}{|x-y|} = \frac{|x^{1/3}-y^{1/3}|}{|x-y|} \to \infty$$

If we let $y=0$, then $\frac{|x^{1/3}|}{|x|} = \frac{1}{|x^{2/3}|} \to \infty$

Thus, we cannot show that there exists a unique local solution.

Complex Numbers:

$\mathbb{C}$ is defined by the numbers defined by $z = \underbrace{x}_{Re(z)} + \underbrace{y}_{Im(z)} i$, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$

A complex number can be represented in polar form $z = r(\cos\theta + i\sin\theta)$

Modulus of $z \rightarrow |z| = \sqrt{x^2+y^2}$

$e^{iz} = \cos z + i\sin z$

$\theta = \operatorname{Arg} z = \arctan \frac{y}{x}$

Ex: Find the modulus and argument of $z = 1+i$, then write z in polar form

$|z| = \sqrt{1+1} = \sqrt{2}$

$\theta = \arg z = \arctan(1) = \frac{\pi}{4}$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \xrightarrow{\text{check!}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \frac{2}{2} + i\frac{2}{2} = 1+i$:)

Week 6 - Oct. 11th

Ex: Midterm Q4. Find a 1-parameter family of solutions to the following equation:

$$xy' + 3y - \sin x^3 = 0$$

Rewrite it $y' + \frac{3}{x}y - \frac{\sin x^3}{x} = 0$ → 1st Order, linear

$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$

$\Rightarrow x^3y' + 3x^2y - x^2\sin x^3 = 0$

$f(x,y) = \int x^3 dy + C(x) = x^3y + C(x)$

$f(x,y) = \int 3x^2y - x^2\sin x^3\, dx + C(y) = x^3y - \frac{1}{3}\cos x^3$

$\therefore x^3y + \frac{1}{3}\cos x^3 = C.$

Second order differential equations:

Homogeneous with constant coefficients

$ay''+by'+cy=0$ where a,b,c are constants.

Characteristic equation $ar^2+br+c=0$, find roots r_1 and r_2

Then $y=c_1e^{r_1t}+c_2e^{r_2t}$ is a general solution

Ex: Find the general solution of $y''+5y'+6y=0$

$r^2+5r+6=0=(r+2)(r+3)=0$
$r_1=-2$ and $r_2=-3$

Then $y=c_1e^{-2t}+c_2e^{-3t}$, $c_1,c_2\in\mathbb{R}$

Ex: Find the solution of the initial value problem $y''+5y'+6y=0$ $y(0)=2$, $y'(0)=3$

$t=0$, $y(0)=c_1+c_2=2$

$y'=-2c_1e^{-2t}-3c_2e^{-3t} \rightarrow y'(0)=-2c_1-3c_2=3$

$\begin{cases} c_1+c_2=2 \\ -2c_1-3c_2=3 \end{cases}$ $-c_2=7 \rightarrow c_2=-7$ and $c_1=9$

$\therefore y=9e^{-2t}-7e^{-3t}$

Ex: Find the solution of the initial value problem $4y''-8y'+3y=0, y(0)=2, y'(0)=\frac{1}{2}$

$4r^2-8r+3=0=(2r-1)(2r-3)$
$r_1=\frac{1}{2}$, $r_2=\frac{3}{2}$

General sol: $y=c_1e^{t/2}+c_2e^{3t/2}$ $y'=\frac{c_1}{2}e^{t/2}+\frac{3c_2}{2}e^{3t/2}$

$\left.\begin{aligned} y(0)&=c_1+c_2=2 \\ y'(0)&=\frac{c_1}{2}+\frac{3c_2}{2}=\frac{1}{2} \end{aligned}\right\}$ $\begin{aligned} c_1&=2-c_2 \\ c_1&=1-3c_2 \end{aligned}$ $c_2=-\frac{1}{2}$, $c_1=\frac{5}{2}$

$\therefore y=\frac{5}{2}e^{t/2}-\frac{1}{2}e^{3t/2}$

★ Ex: Find the solution of $y^{(4)}+y'''-7y''-y'+6y=0$ that satisfies the initial condition $y(0)=1$, $y'(0)=0$, $y''(0)=-2$, $y'''(0)=-1$

$r^4+r^3-7r^2-r+6=0$ Roots are $r_1=1, r_2=-1, r_3=2, r_4=-3$

General solution $y=c_1e^t+c_2e^{-t}+c_3e^{2t}+c_4e^{-3t}$

$y'=c_1e^t-c_2e^{-t}+2c_3e^{2t}-3c_4e^{-3t}$

$y''=c_1e^t+c_2e^{-t}+4c_3e^{2t}+9c_4e^{-3t}$

$y'''=c_1e^t-c_2e^{-t}+8c_3e^{2t}-27c_4e^{-3t}$

$$\left.\begin{aligned} y(0) &= c_1+c_2+c_3+c_4=1 \\ y'(0) &= c_1-c_2+2c_3-3c_4=0 \\ y''(0) &= c_1+c_2+4c_3+9c_4=-2 \\ y'''(0) &= c_1-c_2+8c_3-27c_4=-1 \end{aligned}\right\} \quad c_1=\frac{11}{8},\ c_2=\frac{5}{12},\ c_3=\frac{-2}{3},\ c_4=-\frac{1}{8}$$

$\therefore\ y=\frac{11}{8}e^t+\frac{5}{12}e^{-t}-\frac{2}{3}e^{2t}-\frac{1}{8}e^{-3t}$

Week 7 - October 18th

No tutorial - Reading week

Week 8 - October 25th

Nonhomogeneous Equations : Method of Undetermined Coefficients

$L[y]=y''+p(t)y'+q(t)y=g(t)$

Ex: Find a particular solution of $y''-3y'-4y=3e^{2t}$

We need to find a function Y such that $Y''(t)-3Y'(t)-4Y(t)=3e^{2t}$
Since the exponential function reproduces itself through differentiation we can try
$Y(t)=Ae^{2t}$, where A needs to be determined.

$Y'(t)=2Ae^{2t}$ $\quad Y''(t)=4Ae^{2t}$

$\Rightarrow\ 4Ae^{2t}-6Ae^{2t}-4Ae^{2t}=3e^{2t}$

$\Rightarrow\ -6A=3$

$A=-\frac{1}{2}$

$\therefore\ Y(t)=-\frac{1}{2}e^{2t}$

Ex: Find the particular solution of $y'' - 3y' - 4y = 2\sin t$

Let's try a similar substitution: $Y(t) = A\sin t$

$Y'(t) = A\cos t \qquad Y''(t) = -A\sin t$

$\rightarrow \; -A\sin t - 3A\cos t - 4A\sin t = 2\sin t$

So $-5A = 2$ and $-3A = 0$

↑ uhoh...

Let's try something else: $Y(t) = A\sin t + B\cos t$

$Y'(t) = A\cos t - B\sin t \qquad Y''(t) = -A\sin t - B\cos t$

$\rightarrow \; -A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t$

$(-5A + 3B)\sin t + (-3A - 5B)\cos t = 2\sin t$

$-5A + 3B = 2$

$-3A - 5B = 0 \rightarrow A = -\frac{5B}{3} \nearrow \; \frac{25B}{3} + \frac{9B}{3} = \frac{34B}{3} = 2 \rightarrow B = \frac{3}{17}$

$\rightarrow A = -\frac{5}{17}$

$\therefore Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$

Ex: Find the particular solution of $y'' - 3y' - 4y = -8e^t\cos 2t$

$Y(t) = Ae^t\cos 2t + Be^t\sin 2t$

$Y'(t) = Ae^t\cos 2t - 2Ae^t\sin 2t + Be^t\sin 2t + 2Be^t\cos 2t$

$= (A+2B)e^t\cos 2t + (-2A+B)e^t\sin 2t$

$Y''(t) = (A+2B)e^t\cos 2t - 2(A+2B)e^t\sin 2t + (-2A+B)e^t\sin 2t + 2(-2A+B)e^t\cos t.$

$= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t$

$\rightarrow \; (-10A - 2B)e^t\cos 2t + (2A - 10B)e^t\sin 2t = -8e^t\cos 2t$

$\begin{cases} -10A - 2B = -8 & \rightarrow B = -5A + 4 \\ 2A - 10B = 0 & \rightarrow 2A + 50A - 40 = 0 \rightarrow A = \frac{10}{13}, \; B = \frac{2}{13} \end{cases}$

$\therefore Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$

Ex: Find the particular solution of $y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t\cos 2t$

We can separate these into three parts and get:

$\therefore Y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t.$

Week 9 - November 1st.

The Method of Variation of Parameters:

Used to solve nonhomogeneous 2nd order linear differential equations:

$$y'' + p(t)y' + q(t)y = g(t).$$

The homogeneous solution is $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$

the particular solution is $Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,ds$

where the Wronskian $W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

the general solution is $y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$

Ex: $y'' - 2y' - 3y = 64te^{-t}$.

Characteristic equation $r^2 - 2r - 3 = 0 = (r-3)(r+1) = 0 \Rightarrow r = -1, 3$

$\Rightarrow y_1 = e^{-t}$ and $y_2 = e^{3t}$. let $t_0 = 0$

$$W(y_1, y_2) = \begin{vmatrix} e^{-t} & e^{3t} \\ -e^{-t} & 3e^{3t} \end{vmatrix} = 3e^{2t} + e^{2t} = 4e^{2t}$$

$$* \int_0^t \frac{e^{3s}\cdot 64se^{-s}}{4e^{2s}}\,ds = 16\int_0^t s\,ds = 16\frac{s^2}{2}\Big|_0^t = 8t^2$$

$$* \int_0^t \frac{e^{-s}\cdot 64se^{-s}}{4e^{2s}}\,ds = 16\int_0^t se^{-4s}\,ds = 16\left[-\frac{se^{-4s}}{4}\Big|_0^t - \frac{1}{4}\int_0^t e^{-4s}\,ds\right] = -4te^{-4t} - e^{-4t} - 1$$

$$\Rightarrow Y(t) = -e^{-t}(8t^2) + e^{3t}(4te^{-4t} - e^{-4t}) = -8t^2e^{-t} - 4te^{-t} - e^{-t} - 1$$

$\therefore$ The general solution is $y(t) = c_1 e^{-t} + c_2 e^{3t} + (-8t^2 - 4t - 1)e^{-t} - 1$

Used to solve nonhomogeneous nth order linear differential equations:

$$y^{(n-1)} + \cdots + p(t)y' + q(t)y = g(t).$$

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

Solution: $Y(t) = \sum_{m=1}^{n} y_m(t)\int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\,ds$, $\quad W(t) = ce^{-\int p_1(t)dt}$

Ex: Given that $y_1(t)=e^t$, $y_2(t)=te^t$, $y_3(t)=e^{-t}$ is the solution of the homogeneous equation corresponding to $y'''-y''-y'+y=g(t)$

$$W(t)=W(e^t,te^t,e^{-t})=\begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix}$$

$$=e^t\begin{vmatrix} 1 & t & 1 \\ 1 & t+1 & -1 \\ 1 & t+2 & 1 \end{vmatrix}=e^t[(t+1+t+2)-2t+(t+2-t-1)]=4e^t$$

$$W_1(t)=\begin{vmatrix} 0 & te^t & e^{-t} \\ 0 & (t+1)e^t & -e^{-t} \\ 1 & (t+2)e^t & e^{-t} \end{vmatrix}=-t-(t+1)=-2t-1$$

$$W_2(t)=\begin{vmatrix} e^t & 0 & e^{-t} \\ e^t & 0 & -e^{-t} \\ e^t & 1 & e^{-t} \end{vmatrix}=-(-1-1)=2$$

$$W_3(t)=\begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix}=(t+1)e^{2t}-te^{2t}=e^{2t}$$

$$Y(t)=e^t\int_{t_0}^{t}\frac{g(s)(-2s-1)}{4e^s}\,ds+te^t\int_{t_0}^{t}\frac{g(s)(2)}{4e^s}\,ds+e^{-t}\int_{t_0}^{t}\frac{g(s)e^{2s}}{4e^s}\,ds$$

$$=\frac{1}{4}\int_{t_0}^{t}\left[e^{t-s}(-1+2t-2s)+e^{-t-s}\right]g(s)\,ds$$

Summary of second-order linear homogeneous equations: $ay''+by'+cy=0$

Let r_1 and r_2 be the roots of $ar^2+br+c=0$

* $r_1, r_2 \in \mathbb{R}$: $y=c_1e^{r_1t}+c_2e^{r_2t}$

* $r_1, r_2 \in \mathbb{C}$ where $r_1=\lambda+i\mu$ and $r_2=\lambda-i\mu$: $y=c_1e^{\lambda t}\cos\mu t+c_2e^{\lambda t}\sin\mu t$

Reduction of Order:

Suppose we know one solution $y_1(t)$ of $y''+p(t)y'+q(t)y=0$.
To find a second solution, let $y=v(t)y_1(t)$; then $y'=v'(t)y_1(t)+v(t)y_1'(t)$ and $y''=v''(t)y_1(t)+2v'(t)y_1'(t)+v(t)y_1''(t)$.

Substituting back we get $y_1v''+(2y_1'+py_1)v'+\underbrace{(y_1''+py_1'+qy_1)}_{=0}v=0$

$\Rightarrow y_1v''+(2y_1''+py_1)v'=0$.

Ex: Given that $y_1(t) = t^{-1}$ is a solution of $2t^2y'' + 3ty' - y = 0$, $t>0$ find a fundamental set of solutions.

* $y = v(t)\, t^{-1}$

* $y' = v't^{-1} - vt^{-2}$

* $y'' = v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3}$
$= v''t^{-1} - 2v't^{-2} + 2vt^{-3}$

Substituting:

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} = 0$$
$$\rightarrow 2v''t - 4v' + 4vt^{-1} + 3v' - 3vt^{-1} - vt^{-1} = 0$$
$$\rightarrow 2v''t - v' = 0$$

Let $u = v'$. Then we have $2u't - u = 0$

$$\Rightarrow u = v' = c_1 t^{1/2}$$
$$\Rightarrow v(t) = \frac{2}{3} c_1 t^{3/2} + c_2$$

$\therefore y = t^{-1}\left(\frac{2}{3} c_1 t^{3/2} + c_2\right)$ is the fundamental set of solutions.

Week 10 – November 8th.

Ex: Transform the given equation into a system of first order equations

(a) $u'' + 0.5u' + 2u = 0$

Let $x_1 = u$ and $x_2 = u'$

$$\therefore \begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 0.5x_2 \end{cases}$$

(b) $t^2u'' + tu' + (t^2 - 0.25)u = 0$

Let $x_1 = u$ and $x_2 = u'$

$$\therefore \begin{cases} x_1' = x_2 \\ x_2' = \left(\frac{0.25}{t^2} - 1\right) x_1 - \frac{x_2}{t} \end{cases}$$

(c) $u^{(4)} - u = 0$

Let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, $x_4 = u^{(3)}$

$\therefore x_1' = x_2, x_2' = x_3, x_3' = x_4, x_4' = x_1$

Matrices review

Ex: Given $A = \begin{pmatrix} 3 & 2-i \\ 4+3i & -5+2i \end{pmatrix}$, find A^T and $\bar{A}$

$A^T = \begin{pmatrix} 3 & 4+3i \\ 2-i & -5+2i \end{pmatrix}$ $\quad \bar{A} = \begin{pmatrix} 3 & 2+i \\ 4-3i & -5-2i \end{pmatrix}$

Determinant of Matrix:

$2\times 2 \rightarrow \det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3 (General) $\rightarrow$ if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then

$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Inverse:

$2\times 2 \rightarrow A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Other $\rightarrow [A \mid I] \sim$ Row ops $\sim [I \mid A^{-1}]$

Ex: Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -6 & -5 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -6 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -6 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Ex: Solve the given system of equations or show that there is no solutions

(a) $x_1 \qquad - x_3 = 0$
$3x_1 + x_2 + x_3 = 1$
$-x_1 + x_2 + 2x_3 = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & -1/3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 7/3 \\ 0 & 0 & 1 & -1/3 \end{array}\right] \qquad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 7/3 \\ -1/3 \end{bmatrix}$$

(b) $x_1 + 2x_2 - x_3 = 1$
$2x_1 + x_2 + x_3 = 1$
$x_1 - x_2 + 2x_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & -3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$\therefore$ No solution!

Find eigenvalues of A: Solve $\det(A - \lambda I) = 0$.

Find the eigenvector: For each eigenvalue, find the eigenvectors $(A - \lambda I)\vec{v} = 0$

Ex: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - (-\lambda - 1) + (1 + \lambda)$$

$$= -\lambda(\lambda - 1)(\lambda + 1) + (\lambda + 1) + (\lambda + 1)$$

$$= (\lambda + 1)(-\lambda^2 + \lambda + 1 + 1)$$

$$= (\lambda + 1)(-\lambda + 2)(\lambda + 1)$$

$$= (\lambda + 1)^2(\lambda - 2) \qquad \therefore \lambda_1 = 2, \ \lambda_2 = \lambda_3 = -1$$

$\underline{\lambda_1 = 2}$ $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = \lambda_3 = -1}$ $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\rightarrow x_1 + x_2 + x_3 = 0$

let $x_1 = c_1$ and $x_2 = c_2$

$$x = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\text{so} \Rightarrow x^{(2)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } x^{(3)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Week 11 - November 15th.

Ex: Find the general solution of the system $x' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} x$.

We can write the system in scalar form: $x_1' = 2x_1$ and $x_2' = -3x_2$.

Each can be also written down as $x_1 = c_1 e^{2t}$ and $x_2 = c_2 e^{-3t}$

So
$$x = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

We can define two solutions:

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \quad \text{and} \quad x^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

The Wronskian of these solution is

$$W = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0$$

$\therefore$ $x^{(1)}$ and $x^{(2)}$ form a fundamental set of solutions

Term Test 2 Q2

Ex: Solve the IVP: $x^2 y'' - 2xy' - 4y = \frac{5}{x}$ $y(1) = 2$, $y'(1) = 2$, $x > 0$

Hint $y = x^4$ is a solution to the corresponding homogeneous equation

Assume $y_2 = x^4 \int u\,dx$ is a solution to the homogeneous equation.

$$y_2' = x^4 u + 4x^3 \int u\,dx$$

$$y_2'' = x^4 u' + 4x^3 u + 12x^2 \int u\,dx + 4x^3 u'$$

$$= x^4 u' + 8x^3 u + 12x^2 \int u\,dx$$

Let's sub-in the coefficients!

$$\begin{cases} \to x^2 y_2'' = x^6 u' + 8x^5 u + 12x^4 \int u\,dx \\ \to -2xy_2' = -2x^5 u - 8x^4 \int u\,dx \\ \to -4y_2 = -4x^4 \int u\,dx \end{cases}$$

$\Rightarrow x^6 u' + 8x^5 u + 12x^4 \int u\,dx - 2x^5 u - 8x^4 \int u\,dx - 4x^4 \int u\,dx = 0$

$\Rightarrow x^6 u' + 6x^5 u = 0$

$\Rightarrow xu' = -6u$

$\Rightarrow \frac{u'}{u} = -\frac{6}{x}$

$\Rightarrow \ln u = -6\ln x = \ln x^{-6}$

$u = x^{-6}$

$\therefore y_2 = x^4 \int x^{-6} dx = x^4 \cdot \frac{x^{-5}}{5} = \frac{-1}{5x}$, but if we ignore the coefficient: $y_2 = \frac{1}{x}$

Use variation of Parameters!

★ $y_1 = x^4$ and $y_2 = \frac{1}{x}$

So $y_p = u_1 x^4 + u_2 \frac{1}{x}$

$\to u_1' \cdot x^4 + u_2' \cdot \frac{1}{x} = 0$ $\quad\to u_1' \cdot 4x^3 - u_2' \cdot \frac{1}{x^2} = \frac{5}{x^3}$

$\Rightarrow u_2' = -u_1' x^5$ $\quad\Rightarrow u_1' \cdot 5x^3 = \frac{5}{x^3}$

$\hookrightarrow u_1 = \frac{-1}{5x^5}$ and $u_2 = -\ln x$

$\Rightarrow y_p = -\frac{1}{5x} - \frac{\ln x}{x}$

★ $y = c_1 x^4 + c_2 \frac{1}{x} - \frac{\ln x}{x}$

★ $y' = c_1 \cdot 4x^3 + c_2(-x^{-2}) - \frac{1}{x^2} + \frac{1}{x^2} \ln x$

By checking in the initial condition:

$$\left.\begin{aligned} y(1) &= c_1 + 2 = 2 \\ y'(1) &= 4c_1 - c_1 - 1 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 1 \\ c_2 &= 1 \end{aligned}$$

$$\therefore\ y = x^4 + \frac{1}{x} - \frac{\ln x}{x}$$

P.S. HW3 Q6

Ex: Prove that the solution of the inhomogeneous equation

$x^{(n)} + c_{n-1}x^{(n-1)} + \dots + c_1\dot{x} + c_0x(t) = g(t)$ is given by

$x(t) = x_h(t) + \int_0^t u(t-s)g(s)\,ds$ where $u(0) = \dot{u}(0) = \dots = u^{(n-2)}(0) = 0,\ u^{(n-1)}(0) = 1$

By the principle of superposition, we set $x_h(t) = 0$

So $x(t) = \int_0^t u(t-s)g(s)\,ds$ ★ Using Leibniz Rule ★

$$\dot{x}(t) = \cancel{u(0)g(t)}^{\,0} + \int_0^t \frac{d}{dt}\left[u(t-s)g(s)\right]ds$$

$$\ddot{x}(t) = \cancel{\dot{u}(0)g(t)}^{\,0} + \int_0^t \frac{d^2}{dt^2}\left[u(t-s)g(s)\right]ds$$

$$\vdots$$

$$x^{(n-1)}(t) = \cancel{u^{(n-2)}(0)g(t)}^{\,0} + \int_0^t \frac{d^{n-1}}{dt^{n-1}}\left[u(t-s)g(s)\right]ds$$

$$x^{(n)}(t) = \cancel{u^{(n-1)}(0)g(t)}^{\,g(t)} + \int_0^t \frac{d^n}{dt^n}\left[u(t-s)g(s)\right]ds$$

$\Rightarrow x^{(n)} + c_{n-1}x^{(n-1)} + \dots + c_1\dot{x} + c_0x(t)$

$$= g(t) + \int_0^t g(s)\frac{d^n}{dt^n}u(t-s)\,ds + c_{n-1}\int_0^t g(s)\frac{d^{n-1}}{dt^{n-1}}u(t-s)\,ds + \dots + c_0\int_0^t g(s)\,u(t-s)\,ds$$

$$= g(t) + \int_0^t g(s)\underbrace{\left[\frac{d^n}{dt^n}u(t-s) + c_{n-1}\frac{d^{n-1}}{dt^{n-1}}u(t-s) + \dots + c_0u(t-s)\right]}_{u \text{ is a homogeneous solution}}ds \quad = 0$$

$= g(t)$

$\boxed{}$

Week 12 – November 22nd.

Recall we can write $y''+py'+qy=0$ as a matrix

$$x(t)=\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad x_1(t)=y(t), \quad x_2(t)=y'(t)$$

Then $$x'=\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}=\begin{pmatrix} x_2 \\ -px_2-qx_1 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Ex: $y''+2y'-3y=0$

Char. equation $r^2+2r-3=0$

$(r-1)(r+3)=0$

$r=-3,1$

$\therefore y(t)=c_1e^{t}+c_2e^{-3t}$

If $x'=Ax$, $A=\begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$, the eigenvalues are $1,-3$

Eigenvectors:

* $\lambda_1=1 \rightarrow A=\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} -x_1+x_2 \\ 3x_1-3x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_1=x_2 \quad \text{so } \vec{v}_1=k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* $\lambda_2=-3 \rightarrow A\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 3x_1+x_2 \\ 3x_1+x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x_2=-3x_1 \quad \text{so } \vec{v}_2=k_2\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$\therefore$ So we have $x(t)=c_1e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}+c_2e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Difference Equations:

$y_{n+1}=f(n,y_n)$, $n=0,1,2,\dots$ is called a first order difference equation. It is first order because the value of y_{n+1} depends on the value of y_n but not on earlier values y_{n-1}, y_{n-2}, etc.

$y_{n+1}=f(n,y_n)$ is linear if f is a linear function on y_n.

★ Let's assume $y_{n+1} = f(y_n)$, $n = 0, 1, 2, \dots$

Then we have $y_1 = f(y_0)$

$$y_2 = f(y_1) = f[f(y_0)]$$

$$y_3 = f(y_2) = f^3(y_0)$$

$$\vdots$$

$$y_n = f(y_{n-1}) = f^n(y_0)$$

★ Another option is $y_{n+1} = \rho_n y_n$, $n = 0, 1, 2, \dots$

Then we have $y_1 = \rho_0 y_0$

$$y_2 = \rho_1 y_1 = \rho_1 \rho_0 y_0$$

$$\vdots$$

$$y_n = \rho_{n-1} \rho_{n-2} \cdots \rho_1 \rho_0 y_0$$

★ Another possible equation is $y_{n+1} = \rho y_n$

Then we have $y_1 = \rho y_0$

$$y_2 = \rho y_1 = \rho^2 y_0$$

$$\vdots$$

$$y_n = \rho^n y_0$$

Ex: $y_{n+1} = -0.9 y_n$

$y_n = (-0.9)^n y_0$

when $n \to \infty$, $y_n \to 0$

Ex: $y_{n+1} = (-1)^{n+1} y_n$

$$y_n = ((-1)^{n+1})^n y_0$$

$$= (-1)^{n^2+n} y_0$$

$$= (-1)^{n^2} (-1)^n y_0$$

$\rightarrow$ $y_n = (-1)^n y_0$

when $n \to \infty$, y_n either $-y_0$ or y_0.

Week 13 – November 29th

Complex eigenvalues and eigenvectors

Ex: Solve the following IVP

$$\vec{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \vec{x} \quad , \quad \vec{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

① Find the eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 65 = 3 - 4\lambda + \lambda^2 + 65 = \lambda^2 - 4\lambda + 68 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 272}}{2} = 2 \pm 8i$$

② Find the eigenvectors

$\underline{\lambda_1 = 2+8i}$

$$\begin{pmatrix} 3-2-8i & -13 \\ 5 & 1-2-8i \end{pmatrix} = \begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-8i & -13 \\ 5 & -1-8i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$5x_1 + (-1-8i)x_2 = 0$$

$$5x_1 = (1+8i)x_2 \quad \rightarrow \quad \xi^{(1)} = \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

So for this eigenvalve and eigenvector, its solution is

$$\vec{x}_1(t) = e^{(2+8i)t} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix} = e^{2t} e^{8it} \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= e^{2t}\left(\cos(8t) + i\sin(8t) \right) \begin{pmatrix} 1+8i \\ 5 \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} \cos(8t) - 8\sin(8t) + 8i\cos(8t) + i\sin(8t) \\ 5\cos(8t) + 5i\sin(8t) \end{pmatrix}$$

$$= \underbrace{e^{2t} \begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix}}_{\vec{u}(t)} + i \underbrace{e^{2t} \begin{pmatrix} 8\cos(8t) + i\sin(8t) \\ 5\sin(8t) \end{pmatrix}}_{\vec{v}(t)}$$

$$\vec{u}(t) \quad + \quad i \quad \vec{v}(t)$$

③ The general solution:

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} \cos(8t) + 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 8\cos(8t) + i\sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

④ The particular solution:

$$\vec{x}(0) = c_1\begin{pmatrix} \cos(0) + 8\sin(0) \\ 5\cos(0) \end{pmatrix} + c_2\begin{pmatrix} 8\cos(0) + i\sin(0) \\ 5\sin(0) \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -10 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$\rightarrow 3 = c_1 + 8c_2 \longrightarrow 5 = 8c_2$

$\rightarrow -10 = 5c_1 \longrightarrow c_1 = -2 \qquad c_2 = \frac{5}{8}$

$$\therefore\ \vec{x}(t) = -2e^{2t}\begin{pmatrix} \cos(8t) + 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + \frac{5}{8}e^{2t}\begin{pmatrix} 8\cos(8t) + i\sin(8t) \\ 5\sin(8t) \end{pmatrix}$$

Variation of Parameters

Recall: $y'' + p(t)y' + q(t)y = g(t)$

Let $y_1(t)$ and $y_2(t)$ be the homogeneous solutions for the equation.

The particular solution for the nonhomogeneous is

$$y_p(t) = -y_1\int \frac{y_2 g(t)}{W(y_1, y_2)}\,dt + y_2\int \frac{y_1 g(t)}{W(y_1, y_2)}\,dt$$

Ex: Find the general solution to the following differential equation.

$$y'' - 2y' + y = \frac{e^t}{t^2+1}$$

① Find the homogeneous solution

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \rightarrow r = 1$

$y_h(t) = c_1e^t + c_2te^t \quad \rightarrow y_1(t) = e^t \quad y_2(t) = te^t$

② Find the Wroskian!

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t}$$

③ Find the particular solution

$$y_p(t) = -e^t \int \frac{te^t e^t dt}{e^{2t}(t^2+1)} + te^t \int \frac{e^t e^t}{e^{2t}(t^2+1)} dt$$

$$= -e^t \int \frac{t}{t^2+1} dt + te^t \int \frac{1}{t^2+1} dt$$

$$= -\frac{1}{2} e^t \ln|1+t^2| + te^t \arctan(t)$$

∴ The general solution is :

$$y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} e^t \ln|1+t^2| + te^t \arctan(t)$$